Autoencoder-based global concave optimization for Electrical Impedance Tomography

Bastian Harrach

(joint work with A. Brojatsch and J. Wagner)

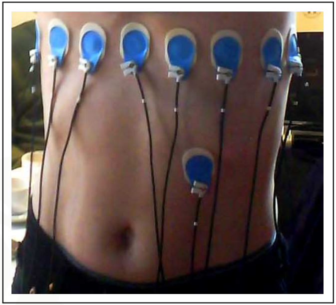
http://numerical.solutions

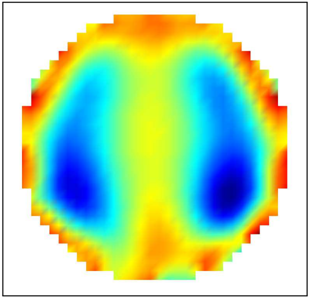
Institute of Mathematics, Goethe University Frankfurt, Germany

Harbin Institute of Technology
March 9–13, 2025

Electrical impedance tomography (EIT)

Electrical impedance tomography (EIT)





- Apply electric currents on subject's boundary
- Measure necessary voltages

$\rightsquigarrow$ Reconstruct conductivity inside subject

Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \subset \mathbb{R}^k \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves (1)}\}\,?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma) : L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

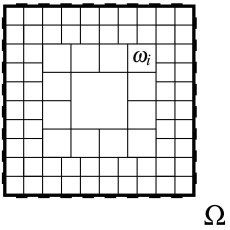
EIT in practice

- Finitely many unknowns, σ pcw. const. on given resolution $\Omega = \bigcup_{i=1}^{n} \omega_i$
- Finitely many measurements

$$\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \,\mathrm{d}s$$

for given currents $g_1, \ldots, g_m \in L^2_\diamond(\partial\Omega)$

(or, analogously, measurements on electrodes)



Finite-dimensional inverse problem: Determine

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix} \in \mathbb{R}^n_+ \quad \text{from} \quad F(\sigma) = \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \,\mathrm{d}s \right)_{j,k=1}^{m} \in \mathbb{R}^{m \times m}.$$

Towards global convergence for practical EIT

Inverse problem: Determine $\sigma \in \mathbb{R}^n_+$ from $Y = F(\sigma) \in \mathbb{R}^{m\times m}$.

- Highly non-linear. Each evaluation of F requires PDE solution.

$\rightsquigarrow$ *Are globally convergent algorithms totally out-of-reach?*

In this talk:

- A concave best data fitting formulation for EIT.
- Globally convergent concave programming in low dimensions.
- Autoencoder for low-dim. parametrization of the image space.
- Using an autoencoder that preserves concavity!

Concave best data fitting formulation for EIT

The Monotonicity Lemma

Lemma. (First appearance: Kang/Seo/Sheen 1997, Ikehata 1998)

$$\int_{\partial\Omega} g(\Lambda(\sigma_1) - \Lambda(\sigma_2))g \, \mathrm{d}s \geq \int_\Omega (\sigma_2 - \sigma_1)|\nabla u_{\sigma_2}^g|^2 \, \mathrm{d}x$$

for all $\sigma_1, \sigma_2 \in L_+^\infty(\Omega)$, $g \in L_\diamond^2(\partial\Omega)$.

- Monotonicity w.r.t. Loewner order:

$$\sigma_1 \leq \sigma_2 \quad \Longrightarrow \quad \Lambda(\sigma_1) \geq \Lambda(\sigma_2)$$

$\rightsquigarrow$ Inclusion detection method (Tamburrino/Rubinacci 2002)

- Localized potentials (**H.** 2008):

$$\exists (g_k)_{k\in\mathbb{N}} : \int_{D_1} |\nabla u_\sigma^{g_k}|^2 \, \mathrm{d}x \to \infty, \quad \int_{D_2} |\nabla u_\sigma^{g_k}|^2 \, \mathrm{d}x \to 0.$$

$\rightsquigarrow$ Converse monotonicity holds for inclusion detection.

$\rightsquigarrow$ Monotonicity method yields exact shape (**H.**/Ullrich 2013).

Monotonicity and Convexity

Lemma.

$$\int_{\partial\Omega} g(\Lambda(\sigma_1)-\Lambda(\sigma_2))g \,\mathrm{d}s \geq \int_{\Omega} (\sigma_2-\sigma_1)|\nabla u^g_{\sigma_2}|^2 \,\mathrm{d}x$$
$$= \int_{\partial\Omega} g\Lambda'(\sigma_2)(\sigma_1-\sigma_2)g \,\mathrm{d}s.$$

for all $\sigma_1, \sigma_2 \in L^\infty_+(\Omega)$, $g \in L^2_\diamond(\partial\Omega)$.

- ⤳ For all $\sigma_1, \sigma_2 \in L^\infty_+(\Omega)$: $\quad \Lambda(\sigma_1)-\Lambda(\sigma_2) \geq \Lambda'(\sigma_2)(\sigma_1-\sigma_2)$.
- ⤳ Convexity: For all $\sigma_1, \sigma_2 \in L^\infty_+(\Omega)$, $t \in [0,1]$

$$\Lambda((1-t)\sigma_1 + t\sigma_2) \leq (1-t)\Lambda(\sigma_1) + t\Lambda(\sigma_2).$$

The "monotonicity lemma" also implies convexity.

- Same results hold for discretized version $F: \mathbb{R}^n_+ \to \mathbb{S}^m \subset \mathbb{R}^{m\times m}$.

Concave residuum funtion

- $F: \mathbb{R}_+^n \to \mathbb{S}^m$ is monotonically non-increasing and convex:

$$\sigma_1 \leq \sigma_2 \quad \Longrightarrow \quad F(\sigma_1) \geq F(\sigma_2)$$
$$F((1-t)\sigma_1 + t\sigma_2) \leq (1-t)F(\sigma_1) + tF(\sigma_2)$$

(for all $\sigma_1, \sigma_2 \in \mathbb{R}_+^n$, $t \in [0,1]$, $\mathbb{S}^m \subset \mathbb{R}^{m\times m}$: symmetric matrices, „$\leq$": Loewner order)

- Best data fitting: If σ fulfills $F(\sigma) = Y$ then σ minimizes

$$\operatorname{trace}(Y - F(\sigma)) \to \min! \quad \texttt{s.t.} \quad F(\sigma) \leq Y.$$

- Residuum functional $\sigma \mapsto \operatorname{trace}(Y - F(\sigma))$ is concave.
- Constraint set $\{\sigma \in \mathbb{R}_+^n : F(\sigma) \leq Y\}$ is convex.

Concave semi-definite minimization over a convex set.

Globally convergent concave programming

Globally convergent concave programming

Hoffman, *Math. Program.* 1981:

Mathematical Programming 20 (1981) 22–32.
North-Holland Publishing Company

A METHOD FOR GLOBALLY MINIMIZING CONCAVE FUNCTIONS OVER CONVEX SETS

Karla Leigh HOFFMAN

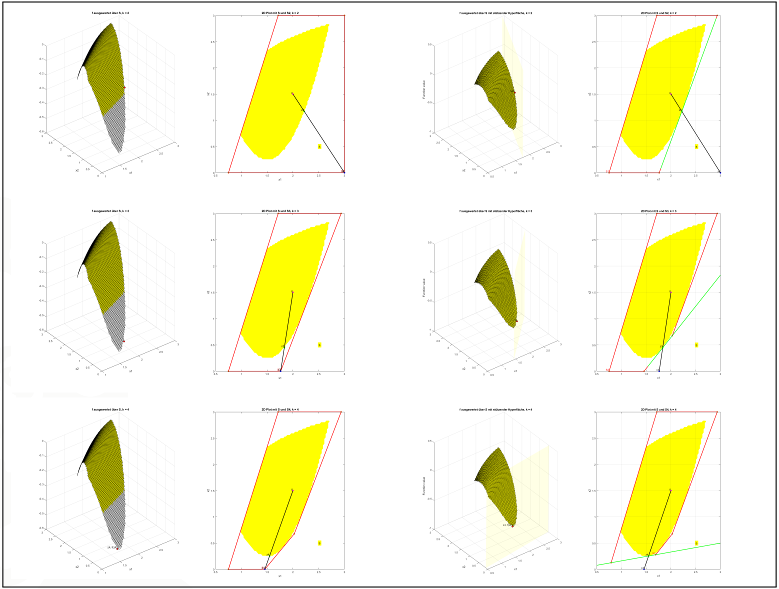
The National Bureau of Standards, Washington, DC, U.S.A.

Received 1 February 1978
Revised manuscript received 17 April 1980

A method is described for globally minimizing concave functions over convex sets whose defining constraints may be nonlinear. The algorithm generates linear programs whose

- Min.s of concave functionals on polyhedra attained in corner.
- For convex sets (compact & generated by finitely many convex inequalities):
  (a) Find polyhedron that containts the constraint set.
  (b) Project best polyhedra corner to constraint set
  (c) Cut out tangential hyperplane in projection point. Repeat (b).

Example (from BSc thesis of N. Lexow)



Global convergence result

Hoffman, *Math. Program.* 1981:

Sequence of best corners converges to global(!) minimizer.

Challenges for our problem

$$\operatorname{trace}(Y - F(\sigma)) \to \min! \quad \texttt{s.t.} \quad F(\sigma) \leq Y.$$

- Result of Hoffman requires compact constraint set.

$\rightsquigarrow$ Add a-priori bounds $\{\sigma \in [a,b]^n : F(\sigma) \leq Y\}$.

- Hoffman's constraint set defined by finitely many inequalities.
- Loewner-order encodes infinitely many scalar inequalities:

$$v^T (Y - F(\sigma))v \geq 0 \;\forall v \in \mathbb{R}^m.$$

$\rightsquigarrow$ Convergence result must be extended (seems possible).

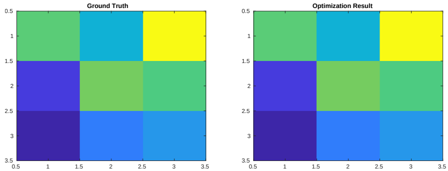
Preliminary numerical result

Globally convergent concave minimization for EIT requires:

- cutting out planes & calculating new corners
- PDE solution for each functional evaluation

(Very preliminary) numerical experiments suggest:

- Global convergence for EIT feasible for low resolutions (e.g. 3×3-image in less than one hour on standard PC).



Global convergence possible for moderately low no. of unknowns.

Autoencoder for low-dim. parametrization

(while preserving concavity!)

Not every image is a lung image

Seo/Kim/Jargal/Lee/**H.** *SIIMS* 2019:

SIAM J. IMAGING SCIENCES
Vol. 12, No. 3, pp. 1275–1295

A Learning-Based Method for Solving Ill-Posed Nonlinear Inverse Problems: A Simulation Study of Lung EIT*

Jin Keun Seo†, Kang Cheol Kim†, Ariungerel Jargal†, Kyounghun Lee†, and Bastian Harrach‡

Abstract. This paper proposes a new approach for solving ill-posed nonlinear inverse problems. For ease of explanation of the proposed approach, we use the example of lung electrical impedance tomography (EIT), which is known to be a nonlinear and ill-posed inverse problem. Conventionally, penalty-based regularization methods have been used to deal with the ill-posed problem. However, experiences over the last three decades have shown methodological limitations in utilizing prior knowledge about tracking expected imaging features for medical diagnosis. The proposed method's paradigm is completely different from conventional approaches; the proposed reconstruction uses a variety of training data sets to generate a low dimensional manifold of approximate solutions, which allows conversion of the ill-posed problem to a well-posed one. Variational autoencoder was used to produce a compact and dense representation for lung EIT images with a low dimensional latent space. Then, we learn a robust connection between the EIT data and the low dimensional latent data. Numerical simulations validate the effectiveness and feasibility of the proposed approach.

- Lung images depend on low number of latent parameters.
- Variational autoencoder finds 16 dim. latent parametrization.
- Inverse problem much more robust to solve on latent space.

Autoencoder for low-dimensional parametrization

Idea: Train neural networks Φ and Ψ so that

$$\Psi \circ \Phi \approx \mathrm{id} \quad \text{on training set of lung images.}$$

- $\Phi: \mathbb{R}^n \to \mathbb{R}^d$ encodes n-pixel images with d latent parameters,
- $\Psi: \mathbb{R}^d \to \mathbb{R}^n$ decodes n-pixel images from d latent parameters.

Inverse Problem: To solve $F(\sigma) = Y$ for lung image σ:

$$\text{Solve} \quad (F \circ \Psi)(p) = Y \quad \text{for } p \in \mathbb{R}^d, \quad \text{then } \Psi(p) = \sigma \in \mathbb{R}^n.$$

Advantages:

- Less unknowns $d << n$, inverse problem more robust.
- $\sigma = \Psi(p)$ lies in range of Ψ (i.e., the set of lung images).
- Data-driven encoding separated from mathematical inversion.

Best data fitting on low-dimensional latent space

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Inverse Problem: To solve $F(\sigma) = Y$ for lung image σ:

$$\text{Solve} \quad (F \circ \Psi)(p) = Y \quad \text{for } p \in \mathbb{R}^d, \quad \text{then } \Psi(p) = \sigma \in \mathbb{R}^n.$$

Best data fitting: If p fulfills $F(\Psi(p))) = Y$ then p minimizes

$$\operatorname{trace}(Y - F(\Psi(p))) \to \min! \quad \texttt{s.t.} \quad F(\Psi(p)) \leq Y.$$

- This is a concave minimization problem on convex set

  if $F \circ \psi : \mathbb{R}^p \to \mathbb{S}^m$ convex (w.r.t. Loewner order).
- Since $F : \mathbb{R}^n \to \mathbb{S}^m$ convex & non-increasing:

  $F \circ \psi : \mathbb{R}^p \to \mathbb{S}^m$ is convex if $\psi : \mathbb{R}^p \to \mathbb{R}^n$ is concave.

$\rightsquigarrow$ *We need low-dim. autoencoder with concave decoder part.*

Enforcing convexity/concavity of neural network

Decoder part Ψ of neural network autoencoder:

- Concatenation of linear functions & activator functions.
- Convex if all parts are convex & non-decrasing.

Simple idea to enforce concavity of Ψ:

- Use convex non-decr. activator functions (e.g., ReLU/Softplus).
- Train NN with non-negativity constraint on all matrices.
- Train with negated images (+ shift), so that $-\Psi$ is convex.

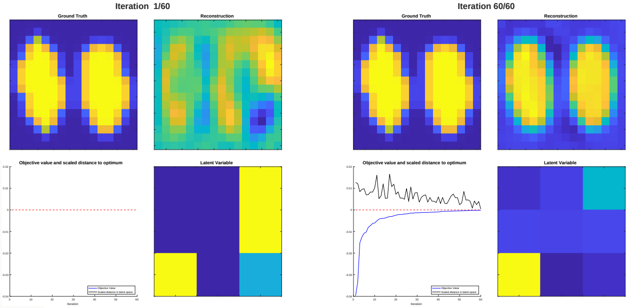
Best data fitting: If $p \in \mathbb{R}^d$ fulfills $F(\Psi(p)) = Y$ then p minimizes

$$\mathrm{trace}(Y - F(\Psi(p))) \to \min! \quad \texttt{s.t.} \quad F(\Psi(p)) \leq Y.$$

Concave min. on convex set in moderately low-dim. latent space.

(Bounds on constraint set in latent space can also be obtained.)

Numerical example



- 2D unit square, 32 electr. (shunt model), 1 grounded, $m = 31$
- $p = 9$ latent variables for 16×16-"lung"-images ($n = 256$)
- Many iterations needed, runtime of several hours ("overnight")

Many open questions

Bottlenecks in our global concave minimization

- Calculating polygon corners from defining inequalites
- Cost function evaluations on each corner (i.e., PDE solutions)
- Better algorithms? DC programming?

Autoencoders with concave encoder

- What image sets can be "well approxmiated"?
- Universal approximation? Conservation of difficulty?
- Bounds in latent space? Ensuring positive conductivities?

Application in EIT

- Ill-posedness? Regularization? Error estimates?

Related question: Learning forward operator

- How to ensure the right mathematical properties?
- For EIT: Monotonicity & convexity w.r.t. Loewner order?

Conclusions

For elliptic coefficient inverse problems

- least-squares residuum functionals may be highly non-convex
- local min. are usually useless, global conv. seems out of reach

Convex best data fitting reformulations

- are possible but some parts still unknown (**H.** SIMA 2023)

Concave best data fitting reformulations

- are easy to derive, all parts known
- allow global convergence in moderately low dimensions

Neural network autoencoder techniques

- parametrize images with low number of latent parameters
- can be trained preserving concavity

Global convergence for images allowing low.-dim. parametrization.