

Inverse Problems in Partial Differential Equations

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March 9–13, 2025

Introduction to inverse problems

Computational Science ("Digital Twin")

Computer simulation of real-world experiment:

Given input x calculate outcome $y = F(x)$.

$x \in X$: parameters / input

$y \in Y$: outcome / measurements

$F : X \rightarrow Y$: functional relation / model

Goals:

- ▶ **Prediction:** Given x , calculate $y = F(x)$.
- ▶ **Optimization:** Find x , such that $F(x)$ is optimal.
- ▶ **Inversion/Identification:** Given $F(x)$, calculate x .

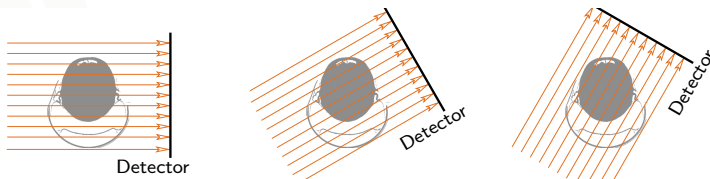
Example: X-ray computerized tomography (CT)

Nobel Prize in Physiology or Medicine 1979:
Allan M. Cormack and Godfrey N. Hounsfield

(Photos: Copyright ©The Nobel Foundation)



Idea: Take x-ray images from several directions



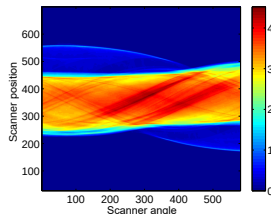
Computerized tomography (CT)

(Image: Hanke-Bourgeois, Grundlagen der Numerischen Mathematik und des Wiss. Rechnens, Teubner 2002)



Image

F



Measurements

Direct problem:

Simulate/predict the measurements

(from knowledge of the interior density distribution)

Given x calculate $F(x) = y!$

Inverse problem:

Reconstruct/image the interior distribution

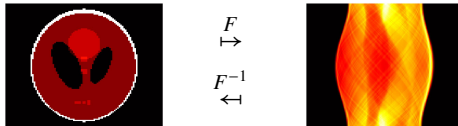
(from taking x-ray measurements)

Given y solve $F(x) = y!$

Computerized tomography

- ▶ CT forward operator $F : x \mapsto y$ is linear
- ↪ Evaluation of F is simple matrix vector multiplication
(after discretizing image and measurements as long vectors)

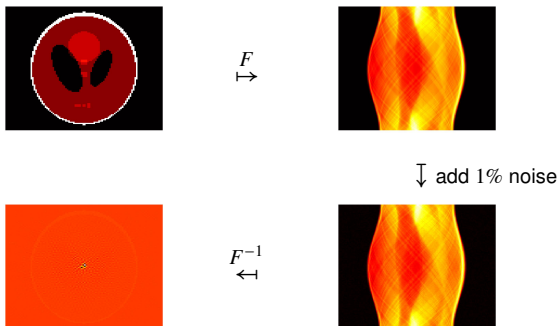
Simple low resolution example:



Problem: Matrix F invertible, but $\|F^{-1}\|$ very large.

Ill-posedness

- ▶ In the continuous case: F^{-1} not continuous
- ▶ After discretization: $\|F^{-1}\|$ very large



Are stable reconstructions impossible?

Ill-posedness

Generic linear ill-posed inverse problem

- ▶ $F : X \rightarrow Y$ bounded and linear, X, Y Hilbert spaces,
- ▶ F injective, F^{-1} not continuous,
- ▶ True solution and noise-free measurements: $F\hat{x} = \hat{y}$,
- ▶ Real measurements: y^δ with $\|y^\delta - \hat{y}\| \leq \delta$

$$F^{-1}y^\delta \not\rightarrow F^{-1}\hat{y} = \hat{x} \quad \text{for} \quad \delta \rightarrow 0.$$

Even the smallest noise may corrupt the reconstructions.

Regularization

Generic linear Tikhonov regularization

$$R_{\alpha} = (F^*F + \alpha I)^{-1}F^*$$

$\leadsto R_{\alpha}$ continuous, $x = R_{\alpha}y^{\delta}$ minimizes

$$\|Fx - y^{\delta}\|^2 + \alpha \|x\|^2 \rightarrow \min!$$

Theorem. Choose $\alpha := \delta$. Then for $\delta \rightarrow 0$,

$$R_{\delta}y^{\delta} \rightarrow F^{-1}\hat{y}.$$

Regularization

Theorem. Choose $\alpha := \delta$. Then for $\delta \rightarrow 0$,

$$R_\delta y^\delta \rightarrow F^{-1} \hat{y}.$$

Proof. Show that $\|R_\alpha\| \leq \frac{1}{\sqrt{\alpha}}$ and apply

$$\|R_\alpha y^\delta - F^{-1} \hat{y}\| \leq \underbrace{\|R_\alpha (y^\delta - \hat{y})\|}_{\leq \|R_\alpha\| \delta} + \underbrace{\|R_\alpha \hat{y} - F^{-1} \hat{y}\|}_{\rightarrow 0 \text{ for } \alpha \rightarrow 0}.$$

Inexact but continuous reconstruction (**regularization**)

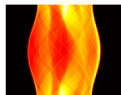
+ Information on measurement noise (**parameter choice rule**)

= Convergence

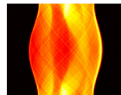
Example ($\delta = 1\%$)



$\hat{x}$



$\hat{y} = F \hat{x}$



y^δ



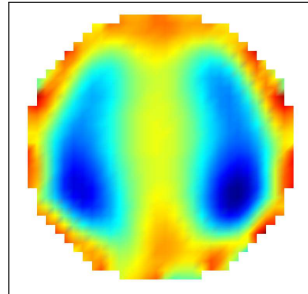
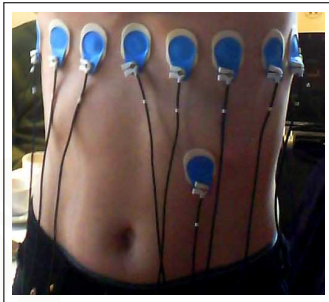
$F^{-1} y^\delta$



$(F^* F + \delta I)^{-1} F^* y^\delta$

Inverse Problems in Partial Differential Equations

Electrical impedance tomography (EIT)



- ▶ Apply electric currents on subject's boundary
- ▶ Measure necessary voltages
- ➞ Reconstruct conductivity inside subject

Calderón problem

Can we recover $\sigma \in L_+^\infty(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \subset \mathbb{R}^d \quad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves (1)}\} ?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma) : L_\diamond^2(\partial\Omega) \rightarrow L_\diamond^2(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

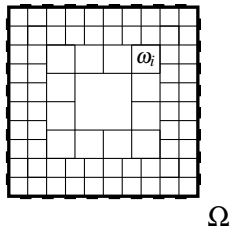
where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

EIT in practice

- ▶ Finitely many unknowns, σ pcw. const. on given resolution $\Omega = \bigcup_{i=1}^n \omega_i$
- ▶ Finitely many measurements

$$\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, ds$$

for given currents $g_1, \dots, g_m \in L^2_{\diamond}(\partial\Omega)$



Finite-dimensional inverse problem: Determine

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix} \in \mathbb{R}_+^n \quad \text{from } F(\sigma) = \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, ds \right)_{j,k=1}^m \in \mathbb{R}^{m \times m}.$$

Simple example: EIT with 2 unknowns & 6 bndry. currents

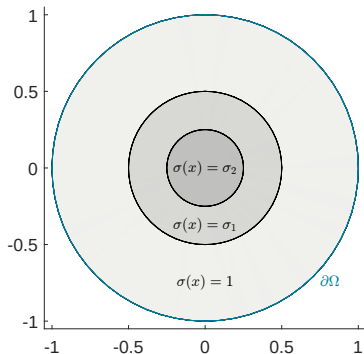
Ω : unit circle

$$F: \mathbb{R}_+^2 \rightarrow \mathbb{R}^{6 \times 6}$$

$$F \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} := \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \right)_{j,k=1}^6$$

with trigonometric currents

$$\{g_1, \dots, g_6\} = \{\sin(\varphi), \dots, \cos(3\varphi)\}$$

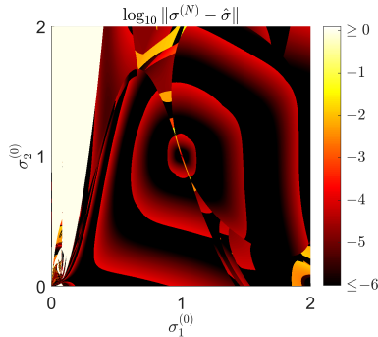


Inverse problem: Reconstruct $\hat{\sigma} \in \mathbb{R}_+^2$ from $\hat{Y} = F(\hat{\sigma}) \in \mathbb{R}^{6 \times 6}$

Natural approach: Least squares data fitting

$$\text{minimize} \quad \|F(\sigma) - \hat{Y}\|_F^2 \quad (+ \text{Regularization})$$

Problem of non-linearity / local convergence



- ▶ Least squares data fitting functional not convex
- ▶ Error of Matlab's `lsqnonlin` highly depends on initial values

Can we derive globally convergent algorithm?

The Monotonicity Lemma

Lemma. (First appearance: Kang/Seo/Sheen 1997, Ikehata 1998)

$$\int_{\partial\Omega} g(\Lambda(\sigma_1) - \Lambda(\sigma_2))g \, ds \geq \int_{\Omega} (\sigma_2 - \sigma_1) |\nabla u_{\sigma_2}^g|^2 \, dx$$

for all $\sigma_1, \sigma_2 \in L_+^\infty(\Omega)$, $g \in L_\diamond^2(\partial\Omega)$.

- Monotonicity w.r.t. Loewner order:

$$\sigma_1 \leq \sigma_2 \implies \Lambda(\sigma_1) \geq \Lambda(\sigma_2)$$

→ Inclusion detection method (Tamburrino/Rubinacci 2002)

- Localized potentials (H. 2008):

$$\exists (g_k)_{k \in \mathbb{N}} : \int_{D_1} |\nabla u_{\sigma}^{g_k}|^2 \, dx \rightarrow \infty, \quad \int_{D_2} |\nabla u_{\sigma}^{g_k}|^2 \, dx \rightarrow 0.$$

→ Converse monotonicity holds for inclusion detection.

→ Monotonicity method yields exact shape (H./Ullrich 2013).

Monotonicity and Convexity

Lemma.

$$\begin{aligned} \int_{\partial\Omega} g(\Lambda(\sigma_1) - \Lambda(\sigma_2))g \, ds &\geq \int_{\Omega} (\sigma_2 - \sigma_1) |\nabla u_{\sigma_2}^g|^2 \, dx \\ &= \int_{\partial\Omega} g\Lambda'(\sigma_2)(\sigma_1 - \sigma_2)g \, ds. \end{aligned}$$

for all $\sigma_1, \sigma_2 \in L_+^\infty(\Omega)$, $g \in L_\diamond^2(\partial\Omega)$.

↷ For all $\sigma_1, \sigma_2 \in L_+^\infty(\Omega)$: $\Lambda(\sigma_1) - \Lambda(\sigma_2) \geq \Lambda'(\sigma_2)(\sigma_1 - \sigma_2)$.

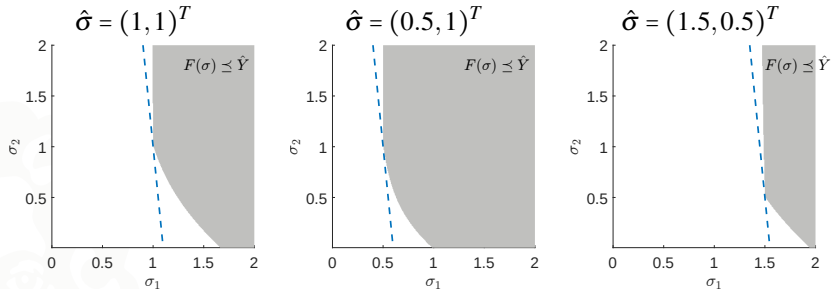
↷ **Convexity:** For all $\sigma_1, \sigma_2 \in L_+^\infty(\Omega)$, $t \in [0, 1]$

$$\Lambda((1-t)\sigma_1 + t\sigma_2) \leq (1-t)\Lambda(\sigma_1) + t\Lambda(\sigma_2).$$

The "monotonicity lemma" also implies convexity.

Convexity for the simple example

Inverse problem: Reconstruct $\hat{\sigma} \in \mathbb{R}_+^2$ from $\hat{Y} = F(\hat{\sigma}) \in \mathbb{R}^{6 \times 6}$

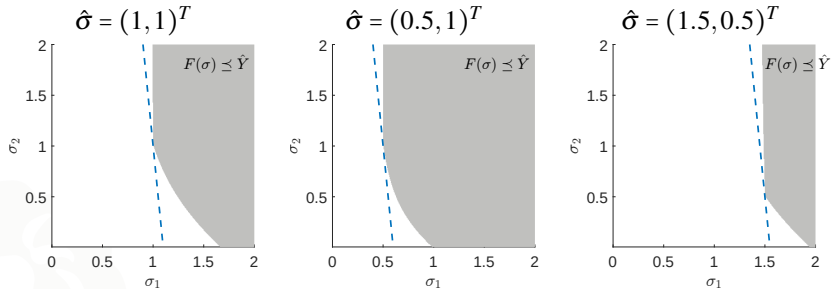


Observation.

$\hat{\sigma}$ is the lower left corner of the convex set $F(\sigma) \preceq \hat{Y}$.

("≤": Loewner / semidefiniteness order)

Mathematical formulation



Conjecture. $\exists c \in \mathbb{R}^n$ so that true solution $\hat{\sigma}$ minimizes

$$c^T \sigma = \sum_{i=1}^n c_i \sigma_i \rightarrow \min! \quad \text{s.t.} \quad \sigma \in [a, b]^n, F(\sigma) \leq \hat{Y}.$$

For a similar but simpler Robin problem:

- ▶ Conjecture holds with $c = \mathbb{1}$ (*H., Optim. Lett. 2021*)
- ▶ Global Newton convergence is possible (*H., Numer. Math. 2021*)

Convex reformulation for EIT

Theorem. (H., SIMA 2023)

If sufficiently many measurements are taken, then:

- ▶ EIT forward mapping $F : [a, b]^n \rightarrow \mathbb{S}_m \subset \mathbb{R}^{m \times m}$ is injective.
- ▶ Derivative $F'(\sigma)$ is injective for all $\sigma \in [a, b]^n$.
- ▶ There exists $c \in \mathbb{R}_+^n$ so that for all $\hat{\sigma} \in [a, b]^n$, $\hat{Y} = \Lambda(\hat{\sigma})$:

$\hat{\sigma}$ is the unique solution of the convex problem

$$\text{minimize } c^T \sigma = \sum_{i=1}^n c_i \sigma_i \quad \text{s.t.} \quad \sigma \in [a, b]^n, F(\sigma) \leq \hat{Y}.$$

The Calderón problem with finitely many unknowns is equivalent to convex semidefinite optimization

Stability and error estimates

Theorem (continued). (H., SIMA 2023)

There exists $\lambda > 0$ so that

- ▶ for all $\hat{\sigma} \in [a, b]^n$, and $\hat{Y} := \Lambda(\hat{\sigma})$,
- ▶ and all $\delta > 0$, and $Y^\delta \in \mathbb{S}_m \subset \mathbb{R}^{m \times m}$, with $\|Y^\delta - \hat{Y}\| \leq \delta$,

the convex semidefinite optimization problem

$$\text{minimize } c^T \sigma = \sum_{i=1}^n c_i \sigma_i \quad \text{s.t.} \quad \sigma \in [a, b]^n, F(\sigma) \leq Y^\delta + \delta I.$$

possesses a minimizer σ^δ . Every such minimizer fulfills

$$\|\sigma^\delta - \hat{\sigma}\|_{c, \infty} \leq \frac{n-1}{\lambda} \delta.$$

($\|\cdot\|_{c, \infty}$: *c-weighted maximum norm*)

Error estimates for noisy data $Y^\delta \approx \hat{Y}$ also hold.

Conclusions

For elliptic coefficient inverse problems

- ▶ least-squares residuum functionals may be highly non-convex
- ▶ local minima are usually useless

Possible remedy

- ▶ utilize monotonicity & convexity with respect to Loewner order

Equivalent convex reformulations are possible

- ▶ globally convergent solution algorithms are possible
- ▶ error estimates for noisy data are possible

Possible extensions?

- ▶ Monotonicity-based methods have been extended to, e.g.,:

Farfield Scattering (Griesmaier/H. 18), Helmholtz (H./Pohjola/Salo 19), Fractional Schrödinger (H./Lin 19+20), Semilinear Schrödinger (Lin 22), Nonlinear Helmholtz (Griesmaier/Knöller/Mandel 22), Waveguide (Furuya 20, Arens/Griesmaier/Zhang 23), Maxwell (Albicker/Griesmaier 23), Crack detection (Furuya 20, Garde/Vogelius 23)