INVERSE PROBLEMS FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. These are the lecture notes for a four lecture series given at the International Summer School of Modern Mathematics "Inverse Problems", Shanghai Center for Mathematical Sciences, June 29 – July 3, 2024. The lectures mainly summarize several works of the author on inverse problems for elliptic PDEs. The outline is as follows.

1. Introduction to inverse problems for elliptic PDEs
2. Monotonicity and localized potentials
3. Monotonicity-based shape reconstruction
4. Uniqueness and Lipschitz stability for finitely many unknowns

During the lecture week, these notes can be downloaded at `https://tinygu.de/SLH24`.

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1. Introduction to inverse problems for elliptic PDEs.

We will introduce the inverse problem of Electrical Impedance Tomography (aka the famous Calderón problem). This problem has received a very high amount of attention as it has important applications in medical imaging and non-destructive testing. It also serves as an example problem since solution methods for EIT can often be extended to other elliptic coefficient problems (e.g. in elasticity, electromagnetics or acoustic scattering).

1.1. Electrical Impedance Tomography (EIT).

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a Lipschitz bounded domain that is subject to stationary (or low-frequency) electrical voltage and current measurements. Let

$$u: \ \Omega \to \mathbb{R},$$

denote the electrical potential function, i.e., $u(x)$ is the voltage in the point $x \in \Omega$ with respect to ground level.

The continuous version of *Ohm's law* states that the current flux resulting from voltage differences is $\sigma(x)\nabla u(x)$, where

$$\sigma: \ \Omega \to \mathbb{R}$$

is the material-dependent conductivity. We assume that the electrical currents can only enter or exit the domain Ω on its boundary, so that

$$\nabla \cdot (\sigma(x)\nabla u(x)) = 0 \quad \text{ for all } x \in \Omega.$$

Non-invasive electrical measurements can only be performed on the boundary $\partial\Omega$. Let $\nu(x)$ denote the outer normal in a boundary point $x \in \partial\Omega$. Then, $u(x)$ is the

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voltage in the boundary point, and $\nu(x) \cdot \sigma(x)\nabla u(x) = \partial_{\nu(x)}\sigma(x)u(x)$ is the electrical current flux that enters or exits the domain at this boundary point.

For the sake of brevity, we usually omit the argument x. We conclude that the inverse problem of EIT is to reconstruct the conductivity coefficient σ in the elliptic PDE

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{ in } \Omega \tag{1.1}$$

from measurements of its Dirichlet boundary values $u|_{\partial\Omega}$ (boundary voltages) and its Neumann boundary values $\sigma\partial_\nu u|_{\partial\Omega}$ (boundary currents).

In the following, we will assume that we drive a certain amount of currents through the imaging domain and measure the required voltages. This means that we prescribe an electrical current flux pattern $g: \ \partial\Omega \to \mathbb{R}$, and measure the required voltages $u|_{\partial\Omega}$, where u solves (1.1) with $\sigma\partial_\nu u|_{\partial\Omega} = g$. We often denote this solution by u_σ^g to highlight its dependance on σ and g.

Note that all results in this lecture series easily carry over to the case of partial boundary measurements, where measurements are only done on an (arbitrarily small) open part $\Sigma \subseteq \partial\Omega$.

1.2. Variational formulation. We assume the reader to be familiar with the concepts of Sobolev spaces, weak derivatives, and the variational formulation of PDEs. Given a conductivity coefficient $\sigma \in L^\infty(\Omega)$, and Neumann boundary data $g \in L^2(\partial\Omega)$, we have that $u \in H^1(\Omega)$ solves

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{ in } \Omega, \quad \text{ and } \quad \sigma\partial_\nu u|_{\partial\Omega} = g \quad \text{ on } \partial\Omega \tag{1.2}$$

in the variational sense, if and only if, $u \in H^1(\Omega)$ solves

$$\int_\Omega \sigma \nabla u \cdot \nabla v \,\mathrm{d}x = \int_{\partial\Omega} gv \,\mathrm{d}s \quad \text{ for all } v \in H^1(\Omega).$$

To comply with the conservation of charge, the amount of currents entering and exiting the domain must sum up to zero. Moreover, the conductivity should be positive, and the solution can only be unique up to a constant function (which corresponds to fixing the ground level of zero voltage). We therefore introduce the spaces

$$\begin{aligned} L^2_\diamond(\partial\Omega) &:= \{g \in L^2(\partial\Omega) : \ \int_{\partial\Omega} g \,\mathrm{d}s = 0\}, \\ H^1_\diamond(\Omega) &:= \{u \in H^1(\Omega) : \ \int_{\partial\Omega} u \,\mathrm{d}s = 0\}, \\ L^\infty_+(\Omega) &:= \{\sigma \in L^\infty(\Omega) : \ \operatorname*{ess\,inf}_{x\in\Omega} \sigma(x) > 0\}. \end{aligned}$$

Note that the ground level fixing in $H^1_\diamond(\Omega)$ is chosen in such a way that the traces of $H^1_\diamond(\Omega)$-functions lie in $L^2_\diamond(\partial\Omega)$.

THEOREM 1.1. *Let $\sigma \in L^\infty_+(\Omega)$, and $g \in L^2_\diamond(\partial\Omega)$. Then $u \in H^1_\diamond(\Omega)$ solves the EIT equation* (1.2) *(in the variational sense), if and only if*

$$\int_\Omega \sigma \nabla u \cdot \nabla v \,\mathrm{d}x = \int_{\partial\Omega} gv \,\mathrm{d}s \quad \textit{for all } v \in H^1_\diamond(\Omega).$$

There exists a unique solution $u_\sigma^g \in H^1_\diamond(\Omega)$, and this solution depends linearly and continuously on g.

Proof. One can show that the H^1-seminorm $u \mapsto \left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)^{1/2}$ is equivalent to the H^1-norm on the subspace $H^1_\diamond(\Omega)$. This yields that

$$b_\sigma:\ H^1_\diamond(\Omega) \times H^1_\diamond(\Omega) \to \mathbb{R}, \quad (u,v) \mapsto \int_\Omega \sigma \nabla u \cdot \nabla v \,\mathrm{d}x$$

is a coercive, continous, and symmetric bilinear form, and

$$l:\ H^1_\diamond(\Omega) \to \mathbb{R}, \quad v \mapsto \int_{\partial\Omega} gv \,\mathrm{d}s$$

is a continuous linear form that depends continuously and linearly on $g \in L^2_\diamond(\partial\Omega)$. Then, the assertion follows from the Lax-Milgram theorem. ☐

1.3. The Neumann-to-Dirichlet operator (NtD). Theorem 1.1 shows that each prescribed electrical current flux pattern g on $\partial\Omega$ uniquely determines the voltage potential u in Ω, which we can then measure on the boundary $\partial\Omega$. Hence, in the idealized setting of the so-called continuum model with infinitely many, infinitely small electrodes, measuring all possible current/voltage-combinations can be modelled by the *Neumann-to-Dirichlet-Operator (NtD)*

$$\Lambda(\sigma):\ L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where $u \in H^1_\diamond(\Omega)$ solves the EIT equation (1.2). By Theorem 1.1, $\Lambda(\sigma)$ is a linear and continuous operator, i.e., $\Lambda(\sigma) \in \mathcal{L}(L^2_\diamond(\partial\Omega))$.

LEMMA 1.2. *For each $\sigma \in L^\infty_+(\Omega)$, the NtD $\Lambda(\sigma) \in \mathcal{L}(L^2_\diamond(\partial\Omega))$ is symmetric, positive definite and compact.*

Proof. Theorem 1.1 yields that

$$\int_{\partial\Omega} g\Lambda(\sigma)h \,\mathrm{d}s = \int_{\partial\Omega} g u_\sigma^h \,\mathrm{d}s = \int_\Omega \sigma \nabla u_\sigma^g \cdot \nabla u_\sigma^h \,\mathrm{d}x,$$

which shows that $\Lambda(\sigma)$ is symmetric and positive definite. The compactness follows from the fact that the trace mapping

$$u \mapsto u|_{\partial\Omega}$$

is a compact linear mapping from $H^1(\Omega)$ to $L^2(\partial\Omega)$. ☐

1.4. Finitely many measurements. In the continuum model with finitely many measurements, we assume that we can only drive finitely many currents

$$\{f_1, \ldots, f_m\} \in L^2_\diamond(\partial\Omega)$$

and that we can only measure the resulting voltage projected to the linear span of these currents. Hence, for $\sigma \in L^\infty_+(\Omega)$, we assume that we can measure the finite-dimensional matrix

$$F_m(\sigma) := \left(\int_{\partial\Omega} f_j \Lambda(\sigma) f_k \,\mathrm{d}s\right)_{j,k=1,\ldots,m} \in \mathbb{R}^{m\times m}.$$

In other words, we assume that the we cannot measure the whole infinite-dimensional NtD, but only its orthogonal (Galerkin) projection to the finite dimensional space $\mathrm{span}\{g_1, \ldots, g_m\}$.

We furthermore make the assumption that that $f_1, \dots, f_m$ are taken from an infinite series $(f_k)_{k\in\mathbb{N}}$, which has dense span in $L^2_\diamond(\partial\Omega)$. This will ensure that $F_m(\sigma)$ approximates the NtD $\Lambda(\sigma)$ when more and more measurements are being used.

In practical applications, one has to model the current flux and voltage measurements on electrodes attached to the imaging object's boundary $\partial\Omega$. The *gap model* assumes that the current flux is constant along each electrode and that one can measure the integral over the voltage over each electrode. This corresponds to the above Galerkin projection with f_j being build up from indicator functions of the electrodes. More realistic electrode model such as the shunt electrode model, and the complete electrode model, do not simply yield a Galerkin projection of the NtD, but they allow similar variational formulations and can be treated widely analogously.

1.5. Differentiability results. For a fixed conductivity σ, the NtD $\Lambda(\sigma)$, and its finitely-many-measurements version $F_m(\sigma)$, are linear operators mapping the applied currents to the measured voltages. But these linear operator depend non-linearly on changes of the conductivity function. The following theorem summarizes the differentiability properties of this non-linear dependence.

THEOREM 1.3. *The mappings*

$$\begin{aligned} \Lambda &: L^\infty_+(\Omega) \to \mathcal{L}(L^2_\diamond(\partial\Omega)), \quad && \sigma \mapsto \Lambda(\sigma),\\ F_m &: L^\infty_+(\Omega) \to \mathbb{R}^{m\times m}, && \sigma \mapsto F_m(\sigma), \end{aligned}$$

are infinitely often Fréchet differentiable. Their first derivatives

$$\Lambda'(\sigma) \in \mathcal{L}(L^\infty(\Omega), \mathcal{L}(L^2_\diamond(\partial\Omega))), \quad \textit{and} \quad F'_m(\sigma) \in \mathcal{L}(L^\infty(\Omega), \mathbb{R}^{m\times m}),$$

are given by

$$\begin{aligned} \int_{\partial\Omega} g\,(\Lambda'(\sigma)d)\,h\,\mathrm{d}s &= -\int_\Omega d\nabla u_\sigma^g \cdot \nabla u_\sigma^h \,\mathrm{d}x,\\ e_j^T\,(F'_m(\sigma)d)\,e_k &= -\int_\Omega d\nabla u_\sigma^{g_j}\cdot\nabla u_\sigma^{g_k}\,\mathrm{d}x \end{aligned}$$

for all $d\in L^\infty(\Omega)$. Here, and in the following, e_j denotes the j-th unit vector in $\mathbb{R}^m$.

Proof. Let

$$\begin{aligned} \langle u,v\rangle_\Omega &:= \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x + \int_\Omega uv\,\mathrm{d}x \quad \forall u,v\in H^1_\diamond(\Omega),\\ \langle g,h\rangle_{\partial\Omega} &:= \int_{\partial\Omega} uv\,\mathrm{d}s \quad \forall g,h\in L^2_\diamond(\Omega) \end{aligned}$$

denote the scalar products on $H^1_\diamond(\Omega)$, and $L^2_\diamond(\partial\Omega)$.

For all $\sigma\in L^\infty_+(\Omega)$, by the Riesz theorem, there exists a unique operator

$$B(\sigma)\in\mathcal{L}(H^1_\diamond(\Omega),H^1_\diamond(\Omega)) \quad \text{with} \quad \langle B(\sigma)u,v\rangle_\Omega = \int_\Omega \sigma\nabla u\cdot\nabla v\,\mathrm{d}x \quad \forall u,v\in H^1_\diamond(\Omega),$$

$B(\sigma)$ is symmetric and coercive, cf. the proof of Theorem 1.1. Hence, by the Lax-Milgram theorem, $B(\sigma)$ is invertible with symmetric inverse

$$B(\sigma)^{-1}\in\mathcal{L}(H^1_\diamond(\Omega),H^1_\diamond(\Omega)).$$

Let $\gamma:\ H^1_\diamond(\Omega)\to L^2_\diamond(\partial\Omega),\ u\mapsto u|_{\partial\Omega}$ denote the trace operator. Then, with these notations, we have that, for all $g\in L^2_\diamond(\partial\Omega)$,

$$\langle B(\sigma)u^g_\sigma, v\rangle = \int_\Omega \sigma\nabla u^g_\sigma\cdot\nabla v\,\mathrm{d}x = \int_{\partial\Omega} g(\gamma v)\,\mathrm{d}s = \langle g,\gamma v\rangle_{\partial\Omega} = \langle\gamma^* g, v\rangle_\Omega,$$

where $\gamma^*:\ L^2_\diamond(\partial\Omega)\to H^1_\diamond(\Omega)$ denotes the adjoint operator of γ.

This shows that $u^g_\sigma = B(\sigma)^{-1}\gamma^* g$, and thus

$$\Lambda(\sigma) = \gamma B(\sigma)^{-1}\gamma^*.$$

We therefore now study the differentiabily properties of $B(\sigma)^{-1}$. Clearly, $B(\sigma)$ depends linearly and continuously on σ, so that $B(\sigma)$ is infinitely often Fréchet differentiable, and

$$\langle (B'(\sigma)d)\,u, v\rangle = \int_\Omega d\,\nabla u\cdot\nabla v\,\mathrm{d}x \quad \text{ for all } d\in L^\infty(\Omega),\ u,v\in H^1_\diamond(\Omega).$$

By the differentiability of operator inversion, we can conclude that $\sigma\mapsto B(\sigma)^{-1}$ is also infinitely often Fréchet differentiable, and that

$$\left(B(\sigma)^{-1}\right)' d = -B(\sigma)^{-1}\left(B'(\sigma)d\right)B(\sigma)^{-1} \quad \text{ for all } d\in L^\infty(\Omega).$$

It thus follows that $\sigma\mapsto\Lambda(\sigma)$ is infinitely often Fréchet differentiable, and

$$\begin{aligned}
\langle g, (\Lambda'(\sigma)d)\,h\rangle_{\partial\Omega} &= \left\langle g, \gamma\left(\left(B(\sigma)^{-1}\right)' d\right)\gamma^* h\right\rangle_{\partial\Omega}\\
&= -\left\langle g, \gamma B(\sigma)^{-1}\left(B'(\sigma)d\right)B(\sigma)^{-1}\gamma^* h\right\rangle_{\partial\Omega}\\
&= -\left\langle \gamma^* g, B(\sigma)^{-1}\left(B'(\sigma)d\right)B(\sigma)^{-1}\gamma^* h\right\rangle_{\Omega}\\
&= -\left\langle \left(B'(\sigma)d\right)B(\sigma)^{-1}\gamma^* h, B(\sigma)^{-1}\gamma^* g\right\rangle_{\Omega}\\
&= -\left\langle \left(B'(\sigma)d\right)u^h_\sigma, u^g_\sigma\right\rangle_{\Omega} = -\int_\Omega d\,\nabla u^g_\sigma\cdot\nabla u^h_\sigma,
\end{aligned}$$

for all $g,h\in L^2_\diamond(\partial\Omega)$, $d\in L^\infty(\Omega)$.

The results are easily carried over to $\sigma\mapsto F_m(\sigma)$ since this is the orthogonal Galerkin projection of $\Lambda(\sigma)$. □

1.6. Some literature and further reading. The term Calderón problem goes back to the famous paper of Calderón [2] (reprinted as [3]). Our differentiability proof follows [7] which also shows how to efficiently implement the matrix-valued measurements $F_m(\sigma)$ in a FEM-setting. For more realistic (extended) electrode models we refer to [14], and for point electrode models see [5].

2. Monotonicity and localized potentials. In this lecture, we introduce two important tools for our further study: the monotonicity and convexity inequality, and the concept of localized potentials.

2.1. The monotonicity and convexity inequality. The following inequality shows that a larger conductivity σ leads to a smaller quadratic form for the NtD $\Lambda(\sigma)$. It also implies convexity of Λ in the so-called Loewner order, we will formulate this in the third lecture.

LEMMA 2.1 (Monotonicity and convexity inequality). *Let $\sigma_1, \sigma_2 \in L^\infty_+(\Omega)$, $g \in L^2_\diamond(\partial\Omega)$, and denote $u_1 := u^g_{\sigma_1}$, and $u_2 := u^g_{\sigma_2}$. Then*

$$\int_\Omega (\sigma_1 - \sigma_2)|\nabla u_2|^2 \,\mathrm{d}x \geq \langle g, (\Lambda(\sigma_2) - \Lambda(\sigma_1))\, g\rangle_{\partial\Omega} \geq \int_\Omega \frac{\sigma_2}{\sigma_1}(\sigma_1 - \sigma_2)|\nabla u_2|^2 \,\mathrm{d}x. \quad (2.1)$$

Proof. Noting that both, u_1 and u_2 solve the EIT equation with the same Neumann data g, we can deduce from Theorem 1.1 that

$$\langle g, \Lambda(\sigma_2) g\rangle_{\partial\Omega} = \int_{\partial\Omega} g u_2 \,\mathrm{d}s = \int_{\partial\Omega} \sigma_1 \partial_\nu u_1\, u_2 \,\mathrm{d}s = \int_\Omega \sigma_1 \nabla u_1 \nabla u_2 \,\mathrm{d}x,$$

and also that

$$\langle g, \Lambda(\sigma_2) g\rangle_{\partial\Omega} = \int_{\partial\Omega} g u_2 \,\mathrm{d}s = \int_{\partial\Omega} \sigma_2 \partial_\nu u_2\, u_2 \,\mathrm{d}s = \int_\Omega \sigma_2 \nabla u_2 \nabla u_2 \,\mathrm{d}x,$$

and analogous expressions hold for $\Lambda(\sigma_1)$. Hence,

$$\begin{aligned}\int_\Omega \sigma_1 |\nabla(u_1 - u_2)|^2 \,\mathrm{d}x &= \int_\Omega \sigma_1 |\nabla u_1|^2 \,\mathrm{d}x - 2\int_\Omega \sigma_2 |\nabla u_2|^2 \,\mathrm{d}x + \int_\Omega \sigma_1 |\nabla u_2|^2 \,\mathrm{d}x \\ &= \langle g, \Lambda(\sigma_1) g\rangle_{\partial\Omega} - \langle g, \Lambda(\sigma_2) g\rangle_{\partial\Omega} + \int_\Omega (\sigma_1 - \sigma_2)|\nabla u_2|^2 \,\mathrm{d}x.\end{aligned}$$

Since the left hand side is non-negative, the first asserted inequality in (2.1) follows.

Interchanging σ_1 and σ_2 we obtain

$$\begin{aligned}\langle g, (\Lambda(\sigma_2) - \Lambda(\sigma_1))\, g\rangle_{\partial\Omega} &= \int_\Omega (\sigma_1 - \sigma_2)|\nabla u_1|^2 \,\mathrm{d}x + \int_\Omega \sigma_2 |\nabla(u_2 - u_1)|^2 \,\mathrm{d}x \\ &= \int_\Omega \left(\sigma_1 |\nabla u_1|^2 + \sigma_2 |\nabla u_2|^2 - 2\sigma_2 \nabla u_1 \cdot \nabla u_2\right) \,\mathrm{d}x \\ &= \int_\Omega \sigma_1 \left|\nabla u_1 - \frac{\sigma_2}{\sigma_1}\nabla u_2\right|^2 \,\mathrm{d}x + \int_\Omega \left(\sigma_2 - \frac{{\sigma_2}^2}{\sigma_1}\right) |\nabla u_2|^2 \,\mathrm{d}x.\end{aligned}$$

Since the first integral on the right hand-side is non-negative, the second asserted inequality follows. □

COROLLARY 2.2. *Interchanging σ_1 and σ_2 in* (2.1) *also yields*

$$\int_\Omega (\sigma_1 - \sigma_2)|\nabla u_1|^2 \,\mathrm{d}x \leq \langle g, (\Lambda(\sigma_2) - \Lambda(\sigma_1))\, g\rangle_{\partial\Omega} \leq \int_\Omega \frac{\sigma_1}{\sigma_2}(\sigma_1 - \sigma_2)|\nabla u_1|^2 \,\mathrm{d}x. \quad (2.2)$$

2.2. Localized potentials. We will show that the energy term $|\nabla u|^2$ in the monotonicity inequality can be made arbitrarily large in some subsection and arbitrarily small in another subsection of Ω provided that the high energy part can be connected to the boundary without crossing the low energy part.

THEOREM 2.3 (Localized potentials). *Let $B \not\subseteq \overline{D}$, where $B, D \Subset \Omega$ are open and $\overline{D}$ has a connected complement. Let $\sigma \in L^\infty_+(\Omega)$ fulfill a unique continuation property*. Then, there exists a sequence $(g_k)_{k\in\mathbb{N}} \subset L^2_\diamond(\partial\Omega)$, so that the corresponding solutions $u_k := u_\sigma^{g_k} \in H^1_\diamond(\Omega)$ fulfill*

$$\int_B |\nabla u_k|^2 \, \mathrm{d}x \to \infty, \qquad \textit{and} \qquad \int_D |\nabla u_k|^2 \, \mathrm{d}x \to 0.$$

The proof of Theorem 2.3 will be given in the next subsection.

REMARK 2.4. *Note that, for the sake of simplicity, we have only present the localized potentials result for the case where the high-energy part does not touch the boundary $\partial\Omega$. More general versions also allow for this, and also hold for partial boundary measurements.*

2.3. Proof of Theorem 2.3.

Reformulation as range (non-)inclusions. To prove Theorem 2.3, we will introduce, for open sets $B, D \subseteq \Omega$, the linear continuous operators

$$L_B : \ L^2_\diamond(\partial\Omega) \to L^2(B)^d, \quad g \mapsto \nabla u_\sigma^g|_B,$$
$$L_D : \ L^2_\diamond(\partial\Omega) \to L^2(D)^d, \quad g \mapsto \nabla u_\sigma^g|_D.$$

Then the assertion of Theorem 2.3 is equivalent to showing that there exists $(g_k)_{k\in\mathbb{N}} \subseteq L^2_\diamond(\partial\Omega)$, so that

$$\|L_B g_k\|_{L^2(B)^d} \to \infty \quad \text{and} \quad \|L_D g_k\|_{L^2(D)^d} \to 0.$$

The norms of operator evaluations are connected to the ranges of the adjoint operators by the following result from functional analysis.

LEMMA 2.5. *Let X, H_1, and H_2 be real Hilbert spaces. Let*

$$A_1 : \ X \to H_1, \qquad \textit{and} \qquad A_2 : \ X \to H_2,$$

be linear bounded operators and denote their adjoint operators with

$$A_1^* : \ H_1 \to X, \qquad \textit{and} \qquad A_2^* : \ H_2 \to X.$$

Then the following holds:

(a) If there exists $C > 0$ with $\|A_1 x\|_{H_1} \le C\|A_2 x\|_{H_2}$ for all $x \in X$, then

$$\mathcal{R}(A_1^*) \subseteq \mathcal{R}(A_2^*).$$

(b) If $\mathcal{R}(A_1^*) \not\subseteq \mathcal{R}(A_2^*)$, then there exists a sequence $(x_k)_{k\in\mathbb{N}}$ with

$$\|A_1 x_k\|_{H_1} \to \infty, \qquad \textit{and} \qquad \|A_2 x_k\|_{H_2} \to 0.$$

*i.e., any solution of the EIT equation which is zero in an open set, or possess zero Dirichlet and Neumann boundary value, must be zero everywhere in Ω

Proof. Let $A_1^* v_1 \in \mathcal{R}(A_1^*) \subseteq X$. Then, for all $x \in X$

$$\langle A_1^* v_1, x\rangle_X = \langle v_1, A_1 x\rangle_{H_1} \leq \|v_1\|_{H_1} \|A_1 x\|_{H_1} \leq C \|v_1\|_{H_1} \|A_2 x\|_{H_2}.$$

This shows that

$$l: \ \mathcal{R}(A_2) \to \mathbb{R}, \quad l(A_2 x) := \langle A_1^* v_1, x\rangle_X$$

is a well-defined, linear and continuous mapping. By unique continuation to $\overline{\mathcal{R}(A_2)}$, and by zero extension on the orthogonal complement $\mathcal{R}(A_2)^\perp$, we can extend l to a linear and continuous functional $l \in \mathcal{L}(H_2, \mathbb{R})$.

Then, by the Riesz theorem, there exists a unique $v_2 \in H_2$ so that

$$\langle v_2, w\rangle_{H_2} = l(w) \quad \text{ for all } w \in H_2.$$

Hence, for all $x \in X$ it follows that

$$\langle A_2^* v_2, x\rangle_X = \langle v_2, A_2 x\rangle_{H_2} = l(A_2 x) = \langle A_1^* v_1, x\rangle_X,$$

which shows that $A_1^* v_1 = A_2^* v_2 \in \mathcal{R}(A_2^*)$. This proves (a).

To prove (b), we note that we obtain from (a) with contraposition that

$$\mathcal{R}(A_1^*) \not\subseteq \mathcal{R}(A_2^*) \quad \Longrightarrow \quad \nexists C > 0: \ \|A_1 x\|_{H_1} \leq C \|A_2 x\|_{H_2} \ \forall x \in X.$$

Hence, for all $k \in \mathbb{N}$, there exists $\xi_k \in X$ with

$$\|A_1 \xi_k\|_{H_1} > k^2 \|A_2 \xi_k\|_{H_2}.$$

If $\|A_2 \xi_k\|_{H_2} > 0$ for all $k \in \mathbb{N}$, then the sequence $x_k := \frac{\xi_k}{k \|A_2 \xi_k\|_{H_2}}$ fulfills

$$\|A_1 x_k\|_{H_1} > k \to \infty, \quad \text{ and } \quad \|A_2 x_k\|_{H_2} = \frac{1}{k} \to 0.$$

Otherwise, if $\|A_2 \xi_j\|_{H_2} = 0$ for some $j \in \mathbb{N}$, we use the sequence $x_k := \frac{k \xi_j}{\|A_1 \xi_j\|_{H_1}}$, and obtain

$$\|A_1 x_k\|_{H_1} = k \to \infty, \quad \text{ and } \quad \|A_2 x_k\|_{H_2} = 0.$$

In both cases, (b) is proven. ☐

Characterization of the adjoints. Using Lemma 2.5, we can deduce Theorem 2.3 from the range (non-)inclusion

$$\mathcal{R}(L_B^*) \not\subseteq \mathcal{R}(L_D^*).$$

We therefore study the adjoints of L_B and L_D.

LEMMA 2.6. *Let $B, D \subseteq \Omega$ be open sets. Then, the adjoints of L_B, and L_D, are given by*

$$L_B^*: \ L^2(B)^d \to L^2_\diamond(\partial\Omega), \quad F \mapsto \nabla v_F|_{\partial\Omega},$$
$$L_D^*: \ L^2(D)^d \to L^2_\diamond(\partial\Omega), \quad G \mapsto \nabla v_G|_{\partial\Omega}.$$

where, for $F \in L^2(B)^d$, and $G \in L^2(D)^d$, the functions $v_F, v_G \in H^1_\diamond(\Omega)$ solve

$$\begin{aligned}
\int_\Omega \sigma \nabla v_F \cdot \nabla w \,\mathrm{d}x &= \int_B F \cdot \nabla w \,\mathrm{d}x, \\
\int_\Omega \sigma \nabla v_G \cdot \nabla w \,\mathrm{d}x &= \int_D G \cdot \nabla w \,\mathrm{d}x.
\end{aligned}$$

Proof. Clearly, L_B is a linear continuous operator, and the existence of v_F follows from the Lax-Milgram theorem. For all $F \in L^2(B)^d$, and $h \in L^2_\diamond(\partial\Omega)$, we have that

$$\begin{aligned}
\langle L_B^* F, h\rangle_{\partial\Omega} = \langle F, L_B h\rangle_B &= \int_B F \cdot \nabla u_\sigma^h \,\mathrm{d}x = \int_\Omega \sigma \nabla v_F \cdot \nabla u_\sigma^h \,\mathrm{d}x \\
&= \int_{\partial\Omega} h v_F|_{\partial\Omega} \,\mathrm{d}s = \langle v_F|_{\partial\Omega}, h\rangle,
\end{aligned}$$

which shows that $L_B^* F = v_F|_{\partial\Omega}$. The same arguments hold for L_D. □

Proof of the range (non-)inclusions. We will now prove the range non-inclusion $\mathcal{R}(L_B^*) \not\subseteq \mathcal{R}(L_D^*)$ under a stronger assumption.

Lemma 2.7. *Let $B, D \Subset \Omega$ be open, let $\overline{B} \cap \overline{D} = \emptyset$, and let $\Omega \setminus (\overline{B} \cup \overline{D})$ be connected. Let σ fulfill a unique continuation property. Then*

$$\mathcal{R}(L_B^*) \cap \mathcal{R}(L_D^*) = \{0\}.$$

Furthermore, if $B \neq \emptyset$, then $\mathcal{R}(L_B^*) \neq \{0\}$, and thus

$$\mathcal{R}(L_B^*) \not\subseteq \mathcal{R}(L_D^*).$$

Proof. Assume that there exists $F \in L^2(B)^d$, and $G \in L^2(D)^d$, with

$$L_B^* F = L_D^* G \in \mathcal{R}(L_B^*) \cap \mathcal{R}(L_D^*).$$

Hence, the corresponding functions $v_F, v_G \in H^1_\diamond(\Omega)$ from Lemma 2.6 fulfill that

$$v_F|_{\partial\Omega} = v_G|_{\partial\Omega}.$$

Moreover, v_F fulfills

$$\int_\Omega \sigma \nabla v_F \cdot \nabla w \,\mathrm{d}x = 0$$

for all $w \in H^1(\Omega)$ with $\operatorname{supp} w \cap \overline{B} = \emptyset$, and an analogous statement holds for v_G. Hence, it follows that $v_F, v_G \in H^1_\diamond(\Omega)$ solve

$$\begin{aligned}
\nabla \cdot (\sigma \nabla v_F) &= 0 \quad \text{in } \Omega \setminus \overline{B}, \quad \text{and} \quad \sigma \partial_\nu v_F|_{\partial\Omega} = 0, \\
\nabla \cdot (\sigma \nabla v_G) &= 0 \quad \text{in } \Omega \setminus \overline{D}, \quad \text{and} \quad \sigma \partial_\nu v_G|_{\partial\Omega} = 0.
\end{aligned}$$

By unique continuation, we obtain that $v_F = v_G$ on $\Omega \setminus (\overline{B} \cup \overline{D})$. Hence, we can define

$$v := \begin{cases} v_F = v_G & \text{on } \Omega \setminus (\overline{B} \cup \overline{D}), \\ v_F & \text{on } \Omega \setminus \overline{B}, \\ v_G & \text{on } \Omega \setminus \overline{D}, \end{cases}$$

to obtain a function $v \in H^1_\diamond(\Omega)$ that solves

$$\nabla \cdot (\sigma \nabla v) = 0 \quad \text{in } \Omega, \quad \text{and} \quad \sigma \partial_\nu v|_{\partial\Omega} = 0.$$

This yields that $v = 0$, and thus $L_B^* F = v_F|_{\partial\Omega} = v|_{\partial\Omega} = 0$. Hence, we have proven that

$$\mathcal{R}(L_B^*) \cap \mathcal{R}(L_D^*) = \{0\}.$$

If $B \neq \emptyset$, then, for all $g \in L^2_\diamond(\partial\Omega)$, unique continuation also yields that

$$L_B g = \nabla u_\sigma^g|_B = 0 \quad \text{implies that} \quad u_\sigma^g = 0 \text{ in } \Omega.$$

Hence, L_B is injective, so that L_B^* has dense range, and thus, a fortiori, $\mathcal{R}(L_B^*) \neq \{0\}$. □

Proof of Theorem 2.3. Now we can prove our localized potentials result. Let $B, D \Subset \Omega$ be open, and let $\overline{B}$ and $\overline{D}$ both have connected complement, and let $B \not\subseteq \overline{D}$. Then, we can find a small ball $\emptyset \neq B' \Subset B \setminus \overline{D}$, and it follows that $\overline{B}' \cap \overline{D} = \emptyset$, and that $\Omega \setminus (\overline{B} \cup \overline{D})$ is connected.

By Lemma 2.7 it follows that

$$\mathcal{R}(L_{B'}^*) \not\subseteq \mathcal{R}(L_D^*).$$

By Lemma 2.5(b), and Lemma 2.6, we thus obtain a sequence $(g_k)_{k\in\mathbb{N}} \subseteq L^2_\diamond(\partial\Omega)$, so that $u_k := u_\sigma^{g_k}$ fulfills

$$\int_{B'} |\nabla u_k|^2 \, \mathrm{d}x = \|L_{B'} g_k\|^2_{L^2(B')^d} \to \infty, \quad \text{and} \quad \int_D |\nabla u_k|^2 \, \mathrm{d}x = \|L_D g_k\|_{L^2(D)^d} \to 0.$$

Of course, this also implies that

$$\int_B |\nabla u_k|^2 \, \mathrm{d}x \geq \int_{B'} |\nabla u_k|^2 \, \mathrm{d}x \to \infty,$$

so that Theorem 2.3 is proven. □

2.4. Some literature and further reading. The monotonicity inequality goes back to [13, 12]. A general version of the functional analytic result connecting norms of operator evaluations to the ranges of the adjoints is listed as the "14th important property of Banach space" in Bourbaki[1]. The idea of localized potentials goes back to [4], our presentation is a simplified variant of [11], where also a more general variant (that allows the high-energy part to touch the boundary $\partial\Omega$) can be found.

3. Monotonicity-based shape reconstruction. We now turn to the inverse problem of reconstructing an unknown conductivity $\sigma \in L^\infty_+(\Omega)$ from the NtD $\Lambda(\sigma) \in \mathcal{L}(L^2_\diamond(\partial\Omega))$ or its finitely-many-measurements version $F_m(\sigma) \in \mathbb{R}^{m\times m}$.

3.1. Loewner monotonicity and convexity. For symmetric operators $A, B \in \mathcal{L}(L^2_\diamond(\partial\Omega))$ we introduce the semidefinite ordering (aka Loewner ordering):

$$A \preceq B \quad \text{denotes that} \quad \int_{\partial\Omega} g(B-A)g \,\mathrm{d}s \geq 0 \quad \text{for all } g \in L^2_\diamond(\partial\Omega).$$

Also, for functions $\sigma, \tau \in L^\infty(\Omega)$,

$$\sigma \leq \tau \quad \text{denotes that} \quad \tau(x) \geq \sigma(x) \text{ for all } x \in \Omega \text{ (a.e.)}$$

THEOREM 3.1. *With respect to these (partial) order relations, the function $\sigma \mapsto \Lambda(\sigma)$ is a convex, monotonically decreasing operator. The following holds:*

(a) For all $\sigma \in L^\infty_+(\Omega)$, and $d \in L^\infty(\Omega)$

$$d \geq 0 \quad \textit{implies} \quad \Lambda'(\sigma)d \preceq 0.$$

For all $\sigma, \tau \in L^\infty_+(\Omega)$,

$$\Lambda(\tau) - \Lambda(\sigma) \succeq \Lambda'(\sigma)(\tau - \sigma).$$

(b) For all $\sigma, \tau \in L^\infty_+(\Omega)$

$$\sigma \geq \tau \quad \textit{implies} \quad \Lambda(\sigma) \preceq \Lambda(\tau).$$

For all $\sigma, \tau \in L^\infty_+(\Omega)$, and $t \in [0,1]$,

$$\Lambda(t\sigma + (1-t)\tau) \preceq t\Lambda(\sigma) + (1-t)\Lambda(\tau).$$

Proof. In Theorem 1.3 we showed that

$$\int_{\partial\Omega} g\,(\Lambda'(\sigma)d)\,h \,\mathrm{d}s = -\int_\Omega d\nabla u^g_\sigma \cdot \nabla u^h_\sigma \,\mathrm{d}x.$$

Hence, the first assertion in (a) is trivial, and the second assertion in (a) is the first monotonicity inequality in Lemma 2.1. Also, the first assertion in (b) immediately follows from the first monotonicity inequality in Lemma 2.1.

It remains to prove the second assertion in (b), which corresponds to Loewner-order convexity. Note that the second assertion in (a) is analogue to the first derivative-based characterization of convexity for a scalar function. Thus, it is easily checked, that the second assertion in (b) follows from the second assertion in (a) exactly as in the scalar case. ∎

The same statements hold for $\sigma \mapsto F_m(\sigma)$ in the finitely-many measurement setting.

3.2. Monotonicity method for inclusion detection. In many applications of EIT, the goal is used to detect regions where the conductivity differs from a know background (inclusion or anomaly detection). A simple example is that $\sigma = 1 + \chi_D$, where $D \subset \Omega$ is the unknown inclusion, and we aim to recover D from measurements $\hat{Y} := \Lambda(1+\chi_D)$. We will explain the monotonicity-based shape reconstruction method for such simple examples, the results are easy to generalize to varying, but known, background conductivities, and to unknown, and varying inclusions contrasts.

A natural monotonicity-based idea is to calculate $\Lambda(1+\chi_B)$ for several small test sets $B \subset \Omega$ (e.g., small balls). By monotonicity, we know that

$$B \subseteq D \quad \text{implies that} \quad \Lambda(1+\chi_B) \succeq \hat{Y}.$$

Hence, if we mark each ball B that fulfills the montonicity test $\Lambda(1+\chi_B) \succeq \hat{Y}$, we can be guaranteed that every ball inside the unknown inclusion gets marked. But, to make this a mathematically rigorous method, we also have to make sure that balls $B \not\subseteq D$ do not get marked. We can prove this using the idea of localized potentials from the last lecture.

Lemma 3.2. *Let $\hat{Y} = \Lambda(1+\chi_D)$ where the inclusion D is open, ∂D is a null set, and $\overline{D} \subset \Omega$ has a connected complement. Then for every open ball $B \subseteq \Omega$*

$$B \subseteq \overline{D} \quad \textit{if and only if} \quad \Lambda(1+\chi_B) \succeq \hat{Y}.$$

In particular, $\overline{D}$ is uniquely determined by knowledge of $\hat{Y} = \Lambda(1+\chi_D)$.

Proof. Since ∂D is a null set, $B \subseteq \overline{D}$ implies that $1+\chi_B \leq 1+\chi_D$ pointwise a.e., so that, by monotonicity, $\Lambda(1+\chi_B) \succeq \Lambda(1+\chi_D) = \hat{Y}$.

It remains to prove that $\Lambda(1+\chi_B) \succeq Y$ implies $B \subseteq \overline{D}$. We show this by contraposition, and assume that $B \not\subseteq \overline{D}$. Let $g_k \in L^2_\diamond(\partial\Omega)$, and $u_k \in H^1_\diamond(\Omega)$ denote the localized potentials from Theorem 2.3 for the conductivity $1+\chi_B$ (which can be shown to fulfill the unique continuity property). Then the first part of the monotonicity inequality (2.1) yields that

$$\begin{aligned}\int_{\partial\Omega} g_k(\Lambda(1+\chi_B) - \hat{Y})g_k \,\mathrm{d}s &\leq \int_\Omega (1+\chi_D) - (1+\chi_B)|\nabla u_k|^2 \,\mathrm{d}x \\ &= \int_D |\nabla u_k|^2 \,\mathrm{d}x - \int_B |\nabla u_k|^2 \,\mathrm{d}x \to -\infty.\end{aligned}$$

This shows that, for sufficiently large $k \in \mathbb{N}$,

$$\int_{\partial\Omega} g_k(\Lambda(1+\chi_B) - \hat{Y})g_k \,\mathrm{d}s < 0,$$

and thus $\Lambda(1+\chi_B) \not\succeq \hat{Y}$. □

3.3. Linearized monotonicity tests and indefinite inclusion detection. A numerical implementation of the monotonicity tests in Lemma 3.2 requires the computation of $\Lambda(1+\chi_B)$, i.e., PDE-solutions, for a large number of test balls B. But our combination of the monotonicity estimate with localized potentials, also allows us to develop the following linearized variant of the monotonicity tests that only require the homogeneous PDE-solution for $\sigma = 1$. Let us stress, that the linearized tests are not merely a numerical approximation, but that they still recover the exact shape.

Lemma 3.3. *Let $\hat{Y} = \Lambda(1+\chi_D)$ where the inclusion D is open, ∂D is a null set, and $\overline{D} \subset \Omega$ has a connected complement. Then for every open set $B \subseteq \Omega$*

$$B \subseteq \overline{D} \quad \textit{if and only if} \quad \Lambda(1) + \frac{1}{2}\Lambda'(1)\chi_B \succeq \hat{Y}.$$

Proof. Using the second inequality in (2.2) we obtain that, for all $g \in L^2_\diamond(\partial\Omega)$,

$$\left\langle g, (\hat{Y} - \Lambda(1) - \tfrac{1}{2}\Lambda'(1)\chi_B)g \right\rangle = \langle g, (\Lambda(1+\chi_D) - \Lambda(1))\, g\rangle - \tfrac{1}{2}\langle g, \Lambda'(1)\chi_B g\rangle$$
$$\leq \int_\Omega \frac{-\chi_D}{1+\chi_D} |\nabla u_1^g|^2 \, \mathrm{d}x + \tfrac{1}{2}\int_B |\nabla u_1^g|^2 \, \mathrm{d}x = -\tfrac{1}{2}\int_D |\nabla u_1^g|^2 \, \mathrm{d}x + \tfrac{1}{2}\int_B |\nabla u_1^g|^2 \, \mathrm{d}x,$$

where u_1^g is the solution of the EIT equation for $\sigma = 1$. Hence,

$$B \subseteq \overline{D} \quad \text{implies} \quad \hat{Y} - \Lambda(1) - \frac{1}{2}\Lambda'(1)\chi_B \preceq 0.$$

To show the converse, we use the first inequality in (2.2) to obtain, for all $g \in L^2_\diamond(\partial\Omega)$,

$$\left\langle g, (\hat{Y} - \Lambda(1) - \tfrac{1}{2}\Lambda'(1)\chi_B)g \right\rangle \geq -\int_D |\nabla u_1^g|^2 \, \mathrm{d}x + \tfrac{1}{2}\int_B |\nabla u_1^g|^2 \, \mathrm{d}x.$$

If $B \not\subseteq \overline{D}$, then we can find a sequence of localized potentials $u_1^{g_k}$, so that

$$\int_D |\nabla u_1^{g_k}|^2 \, \mathrm{d}x \to 0, \quad \text{and} \quad \int_B |\nabla u_1^{g_k}|^2 \, \mathrm{d}x \to \infty.$$

Hence,

$$B \not\subseteq \overline{D} \quad \text{implies} \quad \exists g \in L^2_\diamond(\partial\Omega): \left\langle g, (\hat{Y} - \Lambda(1) - \tfrac{1}{2}\Lambda'(1)\chi_B)g \right\rangle > 0,$$

so that the assertion is proven. □

Unlike other inclusion detection methods, the monotonicity method can also rigorously handle the case of indefinite inclusions, i.e. in a settings where some regions have a higher conductivity than the background, and other regions have a lower conductivity. Again, we demonstrate this on a sample example.

Lemma 3.4. *Let $\hat{Y} = \Lambda(1+\chi_{D^+} - \frac{1}{2}\chi_{D^-})$ where the inclusions $D^+, D^- \subseteq \Omega$ are open, $\overline{D}^+$ and $\overline{D}^-$ are disjoint, and $\overline{D}^+ \cup \overline{D}^- \subset \Omega$ has a connected complement. Then, for every closed $C \subset \Omega$ with null set ∂C and connected complement*

$$\begin{aligned} D^+ \cup D^- \subseteq C \quad & \textit{if and only if} \quad \Lambda(1+\chi_C) \preceq \hat{Y} \preceq \Lambda(1 - \tfrac{1}{2}\chi_C) \\ & \textit{if and only if} \quad \Lambda(1) + \Lambda'(1)\chi_C \preceq \hat{Y} \preceq \Lambda(1) - \Lambda'(1)\chi_C. \end{aligned}$$

Proof. Let $D^+ \cup D^- \subseteq C$. Then

$$1 + \chi_{D^+} - \tfrac{1}{2}\chi_{D^-} \leq 1 + \chi_{D^+} \leq 1 + \chi_C,$$

and thus, by monotonicity, $\Lambda(1+\chi_C) \preceq \hat{Y}$. Moreover, by Theorem 3.1(a),

$$\hat{Y} - \Lambda(1) \succeq \Lambda'(1)(\chi_{D^+} - \tfrac{1}{2}\chi_{D^-}) \succeq \Lambda'(1)\chi_C.$$

Likewise, $D^+ \cup D^- \subseteq C$ also implies that

$$1 - \tfrac{1}{2}\chi_C \leq 1 + \chi_{D^+} - \tfrac{1}{2}\chi_{D^-},$$

so that, by monotonicity, $\Lambda(1-\frac{1}{2}\chi_C) \succeq \hat{Y}$. Morever, by the second inequality in (2.2), for all $g \in L^2_\diamond(\partial\Omega)$,

$$\begin{aligned}\left\langle g, \left(\hat{Y}-\Lambda(1)\right) g\right\rangle &\leq \int_\Omega \left(\frac{-\chi_{D^+}+\frac{1}{2}\chi_{D^-}}{1+\chi_{D^+}-\frac{1}{2}\chi_{D^-}}\right) |\nabla u_1^g|^2 \,\mathrm{d}x \\ &\leq \int_\Omega \left(\frac{\frac{1}{2}\chi_{D^-}}{1+\chi_{D^+}-\frac{1}{2}\chi_{D^-}}\right) |\nabla u_1^g|^2 \,\mathrm{d}x = \int_{D^-} |\nabla u_1^g|^2 \,\mathrm{d}x \\ &\leq \int_C |\nabla u_1^g|^2 \,\mathrm{d}x = -\left\langle g, \Lambda'(1)\chi_C g\right\rangle,\end{aligned}$$

which shows $\hat{Y} \preceq \Lambda(1) - \Lambda'(1)\chi_C$.

Now assume that $D^+ \cup D^- \not\subseteq C$. Then, either $D^+ \not\subseteq C$, or $D^- \not\subseteq C$. In the first case, $D^+ \not\subseteq C$, we can find a sequence of localized potentials $u_1^{g_k}$ for the conductivity $\sigma = 1$ with

$$\int_{D^+} |\nabla u_1^{g_k}|^2 \,\mathrm{d}x \to \infty, \quad \text{and} \quad \int_C |\nabla u_1^{g_k}|^2 \,\mathrm{d}x \to 0.$$

Hence, using the second inequality in (2.2),

$$\begin{aligned}\left\langle g_k, (\hat{Y}-\Lambda(1)-\Lambda'(1)\chi_C) g_k\right\rangle &\leq \int_\Omega \frac{-\chi_{D^+}+\frac{1}{2}\chi_{D^-}}{1+\chi_{D^+}-\frac{1}{2}\chi_{D^-}} |\nabla u_1^{g_k}|^2 \,\mathrm{d}x + \int_C |\nabla u_1^{g_k}|^2 \,\mathrm{d}x \\ &\leq -\frac{1}{2}\int_{D^+} |\nabla u_1^{g_k}|^2 \,\mathrm{d}x + \int_C |\nabla u_1^{g_k}|^2 \,\mathrm{d}x \to -\infty,\end{aligned}$$

which shows that $\hat{Y} \not\succeq \Lambda(1) + \Lambda'(1)\chi_C$. Since, by convexity, $\Lambda(1+\chi_C) \succeq \Lambda(1) + \Lambda'(1)\chi_C$, it also follows that $\hat{Y} \not\succeq \Lambda(1) + \Lambda'(1)\chi_C$.

In the second case, $D^- \not\subseteq C$, we can argue analogously and obtain that $\hat{Y} \not\preceq \Lambda(1-\frac{1}{2}\chi_C)$, and that $\hat{Y} \not\preceq \Lambda(1) - \Lambda'(1)\chi_C$. □

Similar results can be derived for the case of a non-homogeneous (but known) background conductivity, for inclusions with varying conductivity contrast, and for partial boundary data.

3.4. Noisy data and finitely many measurements. The monotonicity tests can be stably implemented in the following sense. Let $A, B \in \mathcal{L}(L^2_\diamond(\partial\Omega))$ be symmetric compact operators for which we want to test whether $A \preceq B$.

We assume that we only now a noisy version of A with $\|A^\delta - A\| < \delta$, where $\delta > 0$ is the noise level, and $\|\cdot\|$ is the spectral norm on $\mathcal{L}(L^2_\diamond(\partial\Omega))$. Then, we implement the *regularized monotonicity tests* and

$$\text{check whether } A^\delta \preceq B + \delta I.$$

- If $A \preceq B$ then the regularized tests will always give the correct answer "yes".
- If $A \not\preceq B$ then, by compactness, $B - A$ possesses a negative eigenvalue $-\lambda < 0$. Hence, for $\delta < \frac{\lambda}{2}$, $A - B \not\preceq 2\delta I$, and thus $A^\delta \not\preceq B + \delta I$, so that the regularized test will give the correct answer "no".

This shows that every monotonicity test can already be decided correctly if the measurement error is below a certain threshold. Moreover, for finitely many measurements, F_m possesses the same monotonicity properties as Λ, and localized potentials can be arbitrarily well approximated if sufficiently many measurements are being used.

Hence, every single monotonicity test can already be correctly decided from noisy finitely many measurements. We summarize this on the example from Lemma 3.3.

LEMMA 3.5. *Let D be an open inclusion with null set ∂D, and let $\overline{D} \subset \Omega$ have a connected complement. Let $Y^\delta \in \mathbb{R}^{m\times m}$ be a symmetric matrix, with*

$$\|Y^\delta - F_m(1+\chi_D)\|_2 < \delta \quad \text{for some noise level } \delta > 0.$$

Then for every open ball $B \subseteq \Omega$:

(a) It holds that

$$B \subseteq \overline{D} \quad \text{implies} \quad F_m(1+\chi_B) \succeq Y^\delta - \delta I.$$

(b) For sufficiently small noise level, and sufficiently many measurements,

$$B \not\subseteq \overline{D} \quad \text{implies} \quad F_m(1+\chi_B) \not\succeq Y^\delta - \delta I.$$

Note that in (b), the allowed noise level and the required number of measurements depend on D on B.

Proof. This can be proven as outlined above. □

3.5. Some literature and further reading. The idea of monotonicty tests was introduced by Tamburrino and Rubinacci [15]. The rigorous justification of the method was given in [11] by proving the converse monotonicity implications. The results in this lecture are a simplified variant of those in [11], where more general variants with spatially varying conductivites in the background and in the inclusions can be found. The fact that linearized monotonictity tests can still recover the exact shape is in line with the results [10] showing that linearization errors do not affect shape reconstructions.

4. Uniqueness and Lipschitz stability for finitely many unknowns. We now turn to the pixel-based setting, where one aims to reconstruct the conductivity only up to a certain resolution.

4.1. The pixel partition. We assume that Ω is decomposed into $n \in \mathbb{N}$ pixels, i.e.,

$$\overline{\Omega} = \bigcup_{j=1}^{n} \overline{P_j}.$$

where $P_1, \ldots, P_n \subseteq \Omega$ are non-empty, pairwise disjoint subdomains with Lipschitz boundaries. We furthermore assume that the pixels are numbered according to their distance from the boundary part Σ, so that the following holds: For any $j \in \{1, \ldots, n\}$ we define

$$Q_j := \bigcup_{i>j} \overline{P_i}$$

and assume that, for all $j = 1, \ldots, n$, the complement of Q_j in $\overline{\Omega}$ is connected and contains a non-empty relatively open subset of Σ.

We will consider conductivity coefficients $\sigma \in L^\infty_+(\Omega)$ that are piecewise constant with respect to this pixel partition, and assume that we know upper and lower bounds, $b > a > 0$. Hence,

$$\sigma(x) = \sum_{j=1}^{n} \sigma_j \chi_{P_j}(x) \quad \text{ with } \sigma_1, \ldots, \sigma_n \in [a, b] \subset \mathbb{R}_+,$$

and $\chi_{P_j} : \Omega \to \mathbb{R}$ denoting the characteristic functions on P_j. In the following, with a slight abuse of notation, we identify such a piecewise constant function $\sigma : \Omega \to \mathbb{R}$ with its coefficient vector $\sigma = (\sigma_1, \ldots, \sigma_n)^T \in \mathbb{R}^n$. Accordingly, we now consider Λ as a non-linear operator

$$\Lambda : \mathbb{R}^n_+ \to \mathcal{L}(L^2_\diamond(\partial\Omega)),$$

and consider the problem to

$$\text{reconstruct} \quad \sigma \in [a, b]^n \subset \mathbb{R}^n_+ \quad \text{from} \quad \Lambda(\sigma) \in \mathcal{L}(L^2_\diamond(\partial\Omega)).$$

Here, and in the following, $\mathbb{R}^n_+ := (0, \infty)^n$ denotes the space of all vectors in $\mathbb{R}^n$ containing only positive entries.

Likewise, we restrict the measurement operator in the case of finitely many measurement, i.e., we consider

$$F : \mathbb{R}^n_+ \to \mathbb{R}^{m \times m},$$

and the (now fully finite-dimensional) inverse problem to

$$\text{reconstruct} \quad \sigma \in [a, b]^n \subset \mathbb{R}^n_+ \quad \text{from} \quad F_m(\sigma) \in \mathbb{R}^{m \times m}.$$

4.2. Monotonicity, convexity, and localized potentials. Our results of the previous lectures clearly also hold for the restriction of Λ to the pixel-based setting. Thus, $\Lambda:\ \mathbb{R}^n_+ \to \mathcal{L}(L^2_\diamond(\partial\Omega))$, and $F_m:\ \mathbb{R}^n_+ \to \mathbb{R}^{m\times m}$, are Fréchet differentiable with continuous derivative

$$\Lambda':\ \mathbb{R}^n_+ \to \mathcal{L}(\mathbb{R}^n, \mathcal{L}(L^2_\diamond(\partial\Omega))), \quad \text{and} \quad F_m':\ \mathbb{R}^n_+ \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^{m\times m}).$$

For all $\sigma \in \mathbb{R}^n_+$, and $d \in \mathbb{R}^n$, the directional derivative $\Lambda'(\sigma)d \in \mathcal{L}(L^2_\diamond(\partial\Omega))$ is compact and selfadjoint, and $F_m'(\sigma)d \in \mathbb{R}^{m\times m}$ is a symmetric matrix.

Moreover, Λ, and F, are both monotonically decreasing, and convex, i.e., for all $\sigma, \tau \in \mathbb{R}^n_+$, and $0 \leq d \in \mathbb{R}^n$,

$$\Lambda'(\sigma)d \preceq 0, \quad \text{and} \quad \Lambda(\tau) - \Lambda(\sigma) \succeq \Lambda'(\sigma)(\tau - \sigma), \tag{4.1}$$

$$F_m'(\sigma)d \preceq 0, \quad \text{and} \quad F_m(\tau) - F_m(\sigma) \succeq F_m'(\sigma)(\tau - \sigma). \tag{4.2}$$

We can also write the localized potentials result in a pixel-based version. For this, we introduce the following notation. As before, the j-th unit vector is denoted by $e_j \in \mathbb{R}^n$. We write $\mathbb{1} := (1, 1, \dots, 1)^T \in \mathbb{R}^n$ for the vector containing only ones, and we write $e_j' := \mathbb{1} - e_j$ for the vector containing ones in all entries except the j-th. We furthermore split $e_j' = e_j^+ + e_j^-$, where

$$e_j^+ := \sum_{i=j+1,\dots,n} e_i, \quad \text{and} \quad e_j^- := \sum_{i=1,\dots,j-1} e_i.$$

Note that we use the usual convention of empty sums being zero, so that $e_n^+ = 0 \in \mathbb{R}^n$, and $e_1^- = 0 \in \mathbb{R}^n$.

Lemma 4.1. *For all $C > 0$, $j \in \{1, \dots, n\}$, and $\sigma \in \mathbb{R}^n_+$, it holds that*

$$\Lambda'(\sigma)(e_j - Ce_j^+) \not\preceq 0, \tag{4.3}$$

and, for sufficiently large $m \in \mathbb{N}$, it also holds that

$$F_m'(\sigma)(e_j - Ce_j^+) \not\preceq 0. \tag{4.4}$$

Proof. By Theorem 1.3, we know that, for all $g \in L^2_\diamond(\partial\Omega)$,

$$\begin{aligned}\left\langle g, \Lambda'(\sigma)(e_j - Ce_j^+)g \right\rangle &= -\int_\Omega (\chi_{P_j} - C\chi_{Q_j}) |\nabla u_\sigma^g|^2 \,\mathrm{d}x \\ &= C\int_{Q_j} |\nabla u_\sigma^g|^2 \,\mathrm{d}x - \int_{P_j} |\nabla u_\sigma^g|^2 \,\mathrm{d}x.\end{aligned}$$

Using a slightly more general version of Theorem 2.3, cf. the references in subsection 2.4, one can show that there exists a sequence $(g_k)_{k\in\mathbb{N}}$ so that the corresponding solutions $u_k := u_\sigma^{g_k}$ fulfill

$$\int_{Q_j} |\nabla u_\sigma^g|^2 \,\mathrm{d}x \to 0, \quad \text{and} \quad \int_{P_j} |\nabla u_\sigma^g|^2 \,\mathrm{d}x \to \infty.$$

Hence, we can find $g \in L^2_\diamond(\partial\Omega)$ with

$$\left\langle g, \Lambda'(\sigma)(e_j - Ce_j^+)g \right\rangle < 0,$$

so that (4.3) is proven.

Recall that the sequence $(f_k)_{k\in\mathbb{N}}$ in the definition of F_m was assumed to have dense span in $L^2_\diamond(\partial\Omega)$. Hence, we can approximate g with a finite linear combination $\tilde{g} = \sum_{k=1}^m v_k f_k$, $v_k \in \mathbb{R}$, and obtain (for sufficiently large $m \in \mathbb{N}$)

$$\left\langle \tilde{g}, \Lambda'(\sigma)(e_j - Ce_j^+)\tilde{g} \right\rangle < 0.$$

Writing $v = (v_1, \dots, v_m)^T \in \mathbb{R}^m$, it follows that

$$v^T F'(\sigma)(e_j - Ce_j^+)v = \left\langle \tilde{g}, \Lambda'(\sigma)(e_j - Ce_j^+)\tilde{g} \right\rangle < 0,$$

so that (4.4) is also proven. □

4.3. Uniqueness and Lipschitz stability for finitely many unknowns. For our fixed pixel partition, and known bounds $b > a > 0$, we will now prove unique solvability and Lipschitz stability for the inverse problem of reconstructing $\sigma \in [a, b]^n \subset \mathbb{R}^n_+$ from continuous data $\Lambda(\sigma)$, and from finitely many measurements $F_m(\sigma)$.

Theorem 4.2. *There exists $c > 0$ such that*

$$\|\Lambda(\sigma) - \Lambda(\tau)\|_{\mathcal{L}(L^2_\diamond(\Sigma))} \geq c\|\sigma - \tau\| \quad \textit{for all } \sigma, \tau \in [a, b]^n.$$

For sufficiently many measurements $m \in \mathbb{N}$, there also exists $c > 0$ such that

$$\|F_m(\sigma) - F_m(\tau)\|_{\mathcal{L}(L^2_\diamond(\Sigma))} \geq c\|\sigma - \tau\| \quad \textit{for all } \sigma, \tau \in [a, b]^n.$$

Note that this shows that for all pixel partitions, i.e., for any fixed desired resolution, $\Lambda(\sigma) \in \mathcal{L}(L^2_\diamond(\Sigma))$ always uniquely determines $\sigma \in \mathbb{R}^n_+$. It also shows that there exists a finite number of measurements, so that $F_m(\sigma)$ uniquely determines $\sigma \in [a, b]^n$, but the number of required number of measurements will depend on the pixel partition, the number of pixels, and the a-priori bounds $b > a > 0$.

Proof. In the following, $\langle \cdot, \cdot \rangle$, always denotes the $L^2_\diamond(\partial\Omega)$-scalar product. We will prove Theorem 4.2 in three steps.

(a) We first use the estimate (4.1) to bound the difference of the non-linear Neumann-to-Dirichlet operators by an expression containing their linearized counterparts. The Neumann-to-Dirichlet-operators are self-adjoint so that for all $\sigma, \tau \in \mathbb{R}^n_+$

$$\|\Lambda(\sigma) - \Lambda(\tau)\| = \sup_{g\in L^2_\diamond(\partial\Omega),\, \|g\|=1} |\langle g, (\Lambda(\sigma) - \Lambda(\tau))\, g\rangle| \,.$$

We apply (4.1), also with interchanged roles of σ and τ, to estimate this expression, and obtain that for all $\sigma, \tau \in \mathbb{R}^n_+$, $\sigma \not\equiv \tau$, and all $g \in L^2_\diamond(\partial\Omega)$

$$\begin{aligned}
&|\langle g, (\Lambda(\sigma) - \Lambda(\tau))\, g\rangle| \\
&= \max\{\langle g, (\Lambda(\tau) - \Lambda(\sigma))\, g\rangle \,, \langle g, (\Lambda(\sigma) - \Lambda(\tau))\, g\rangle\} \\
&\geq \max\{\langle \Lambda'(\sigma)(\tau - \sigma)g, g\rangle \,, \langle \Lambda'(\tau)(\sigma - \tau)g, g\rangle\} \\
&= \|\sigma - \tau\| \max\left\{\left\langle \Lambda'(\sigma)\frac{\tau - \sigma}{\|\sigma - \tau\|}g, g\right\rangle, \left\langle \Lambda'(\tau)\frac{\sigma - \tau}{\|\sigma - \tau\|}g, g\right\rangle\right\} \\
&= \|\sigma - \tau\|\, f(\sigma, \tau, \frac{\tau - \sigma}{\|\sigma - \tau\|}, g),
\end{aligned}$$

where $f:\ [a,b]^n \times [a,b]^n \times \mathcal{K} \times L^2_\diamond(\Sigma) \to \mathbb{R}$ is defined by

$$f(s,t,\kappa,g) := \max\left\{\langle(\Lambda'(s)\kappa)\, g, g\rangle\, , -\langle(\Lambda'(t)\kappa)\, g, g\rangle\right\},$$

and $\mathcal{K} := \{\kappa \in \mathbb{R}^n :\ \|\kappa\| = 1\}$.
Hence, for all $\sigma, \tau \in [a,b]^n$,

$$\begin{aligned}\frac{\|\Lambda(\sigma)-\Lambda(\tau)\|}{\|\sigma-\tau\|} &= \sup_{g\in L^2_\diamond(\partial\Omega),\, \|g\|=1} \frac{|\langle g, (\Lambda(\sigma)-\Lambda(\tau))\, g\rangle|}{\|\sigma-\tau\|}\\ &\geq \sup_{g\in L^2_\diamond(\partial\Omega),\, \|g\|=1} f(\sigma,\tau,\frac{\tau-\sigma}{\|\sigma-\tau\|},g)\\ &\geq \inf_{\substack{(s,t,\kappa)\\ \in [a,b]^n\times[a,b]^n\times\mathcal{K}}} \sup_{g\in L^2_\diamond(\partial\Omega),\, \|g\|=1} f(s,t,\kappa,g).\end{aligned}$$

(b) The continuity of Λ' implies that also f is continuous. Hence, the function

$$(s,t,\kappa) \mapsto \sup_{g\in L^2_\diamond(\Sigma),\, \|g\|=1} f(s,t,\kappa,g)$$

is lower semicontinuous and thus attains its minimum over the compact set $[a,b]^n \times [a,b]^n \times \mathcal{K}$.
Hence, there exists $(\hat\sigma,\hat\tau,\hat\kappa) \in [a,b]^n \times [a,b]^n \times \mathcal{K}$ so that

$$\inf_{\substack{(s,t,\kappa)\\ \in [a,b]^n\times[a,b]^n\times\mathcal{K}}} \sup_{g\in L^2_\diamond(\Sigma),\, \|g\|=1} f(s,t,\kappa,g) = \sup_{g\in L^2_\diamond(\Sigma),\, \|g\|=1} f(\hat\sigma,\hat\tau,\hat\kappa,g),$$

and thus

$$\|\Lambda(\sigma)-\Lambda(\tau)\| \geq \|\sigma-\tau\| \sup_{g\in L^2_\diamond(\Sigma),\, \|g\|=1} f(\hat\sigma,\hat\tau,\hat\kappa,g).$$

(c) It only remains to show that

$$\sup_{g\in L^2_\diamond(\Sigma),\, \|g\|=1} f(\hat\sigma,\hat\tau,\hat\kappa,g) > 0. \tag{4.5}$$

To prove this, let $\hat\kappa_1,\dots,\hat\kappa_n \in [-1,1]$ denote the entries of $\hat\kappa \in \mathcal{K} \subset \mathbb{R}^n$, and let $j \in \{1,\dots,n\}$ denote the index of the first non-zero entry. We now distinguish the two cases, $\hat\kappa_j > 0$, and $\hat\kappa_j < 0$:

(i) If $\hat\kappa_j > 0$, then $\hat\kappa \geq \hat\kappa_j e_j - e_j^+$. Using Lemma 4.1, we have that

$$\exists g \in L^2_\diamond(\Sigma):\quad \left\langle g, \Lambda'(\hat\tau)(e_j - \frac{1}{\hat\kappa_j} e_j^+) g\right\rangle < 0,$$

so that we obtain, using (4.1),

$$\begin{aligned} f(\hat\sigma,\hat\tau,\hat\kappa,g) &\geq -\langle(\Lambda'(\hat\tau)\hat\kappa)\, g, g\rangle \geq -\left\langle\left(\Lambda'(\hat\tau)\left(\hat\kappa_j e_j - e_j^+\right)\right) g, g\right\rangle\\ &= -\frac{1}{\hat\kappa_j}\left\langle\left(\Lambda'(\hat\tau)\left(e_j - \frac{1}{\hat\kappa_j}e_j^+\right)\right) g, g\right\rangle > 0.\end{aligned}$$

(ii) If $\hat\kappa_j < 0$, then $\hat\kappa \leq \hat\kappa_j e_j + e_j^+$. Then, again, by Lemma 4.1

$$\exists g \in L^2_\diamond(\Sigma):\quad \left\langle g, \Lambda'(\hat\sigma)(e_j + \frac{1}{\hat\kappa_j} e_j^+) g\right\rangle < 0,$$

so that we obtain, using (4.1),

$$\begin{aligned} f(\hat\sigma,\hat\tau,\hat\kappa,g) &\geq \langle (\Lambda'(\hat\sigma)\hat\kappa)\, g, g\rangle \geq \left\langle \left(\Lambda'(\hat\sigma)\left(\hat\kappa_j e_j + e_j^+\right)\right) g, g\right\rangle \\ &= \frac{1}{\hat\kappa_j}\left\langle \left(\Lambda'(\hat\tau)\left(e_j + \frac{1}{\hat\kappa_j} e_j^+\right)\right) g, g\right\rangle > 0. \end{aligned}$$

Hence, in both cases, (4.5), and thus the first assertion of Theorem 4.2 is proven. The second assertion follows with the same proof by replacing $\Lambda(\sigma)$ with $F_m(\sigma)$. □

4.4. Outlook: Reformulation as a convex semidefinite program. The results in this lecture show that (with sufficiently many measurements), $\hat Y = F_m(\hat\sigma)$ uniquely determines $\hat\sigma \in [a,b]^n$. A natural approach to solve this inverse problem is to minimize the least squares residual functional

$$\texttt{minimize} \quad \|\hat Y - F_m(\sigma)\|_{\mathrm{F}}^2 \to \min! \quad \texttt{subject to} \quad \sigma \in [a,b]^n.$$

or a regularized variant thereof. However, this functional is non-convex, and numerical algorithms based on this strategy usually suffer from only local convergence.

It is therefore desirable to utilize the Loewner convexity properties to find convex reformulations of the EIT problem. A recent result shows that, for sufficiently many measurements, there exists $c \in \mathbb{R}_+^n$, so that for all $\hat\sigma \in [a,b]^n$, and $\hat Y := F_m(\hat\sigma)$, the minimization problem

$$\texttt{minimize} \quad c^T\sigma \to \min! \quad \texttt{subject to} \quad \sigma \in [a,b]^n,\ F_m(\sigma) \preceq \hat Y, \tag{4.6}$$

possesses a unique minimizer and this minimizer is $\hat\sigma$. Note that this is a linear minimization problem under a convex non-linear semidefinite constraint.

4.5. Some literature and further reading. This lecture is based on [6], where the result is also proven for a more realistic electrode model. The reformulation as a convex semidefinte program is shown in [9], where also the case of noisy measurements is treated. We also refer to [8] for an easy-to-read variant of the convex semidefinite reformulation for a related but simpler inverse Robin coefficient problem.

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