Monotonicity and Convexity in inverse coefficient problems

Bastian Harrach

http://numerical.solutions

Institute of Mathematics, Goethe University Frankfurt, Germany

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Electrical impedance tomography (EIT)

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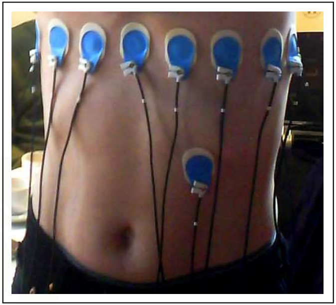
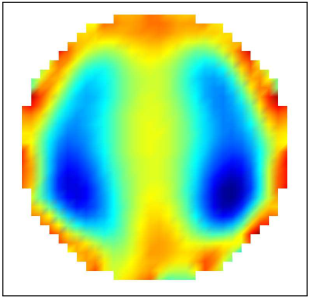




- Apply electric currents on subject's boundary
- Measure necessary voltages

⤳ Reconstruct conductivity inside subject

Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \subset \mathbb{R}^d \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma\partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves (1)}\}\,?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma):\ L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma\partial_\nu u|_{\partial\Omega} = g$.

Challenges in idealized EIT

Mathematical idealization of EIT $\rightsquigarrow$ Calderón problem

- infinitely many unknowns $\sigma \in L^\infty_+(\Omega)$
- infinitely many measurements $\Lambda(\sigma) \in \mathcal{L}(L^2_\diamond(\partial\Omega))$
- nonlinear forward map $\sigma \mapsto \Lambda(\sigma)$

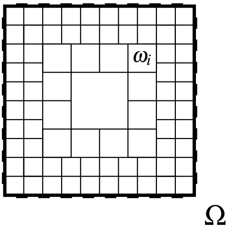
Mathematical challenges

- Uniqueness? Does $\Lambda(\sigma)$ determine σ?
- Stability? $\Lambda^{-1}: \Lambda(\sigma) \mapsto \sigma$ continuous?
- Convergence (local/global)? How to determine σ from $\Lambda(\sigma)$?

Consequences for practical EIT?

EIT in practice

- Finitely many unknowns, σ pcw. const. on given resolution $\Omega = \bigcup_{i=1}^{n} \omega_i$
- Finitely many measurements
$$\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, \mathrm{d}s$$
for given currents $g_1, \ldots, g_m \in L^2_\diamond(\partial\Omega)$



Finite-dimensional inverse problem: Determine

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix} \in \mathbb{R}^n_+ \quad \text{from } F(\sigma) = \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, \mathrm{d}s \right)_{j,k=1}^{m} \in \mathbb{R}^{m\times m}.$$

Mathematical challenges for practical EIT

Inverse problem: Determine $\sigma \in \mathbb{R}_+^n$ from $Y = F(\sigma) \in \mathbb{R}^{m\times m}$.

For a fixed desired resolution:

- How many measurements uniquely determine σ?
- Stability / error estimates for noisy data $Y^\delta \approx F(\sigma)$?
- Numerical algorithm to determine $\sigma \in \mathbb{R}_+^n$ from $Y^\delta \approx F(\sigma)$?
- Global/local convergence of algorithm?

Next slides: The problem of local convergence

Simple example: EIT with 2 unknowns & 6 bndry. currents

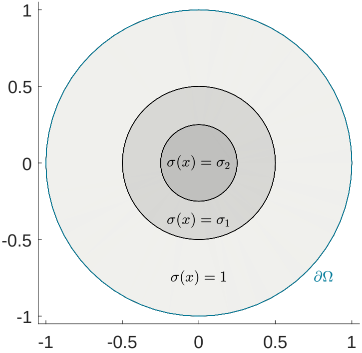
Ω: unit circle

$$F: \mathbb{R}_+^2 \to \mathbb{R}^{6\times 6}$$

$$F\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} := \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k\right)_{j,k=1}^{6}$$

with trigonometric currents

$$\{g_1,\ldots,g_6\} = \{\sin(\varphi),\ldots,\cos(3\varphi)\}$$

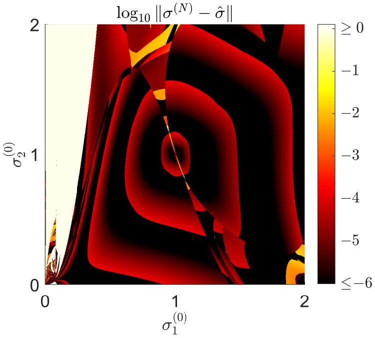


Inverse problem: Reconstruct $\hat{\sigma} \in \mathbb{R}_+^2$ from $\hat{Y} = F(\hat{\sigma}) \in \mathbb{R}^{6\times 6}$

Natural approach: Least squares data fitting

$$\texttt{minimize} \quad \|F(\sigma) - \hat{Y}\|_{\mathsf{F}}^2 \quad \textit{(+ Regularization)}$$

Problem of local convergence



- Least squares data fitting functional not convex
- Error of Matlab's `lsqnonlin` highly depends on initial values

Can we derive globally convergent algorithms?

Monotonicity and Convexity

The Monotonicity Lemma

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Lemma. (First appearance: Kang/Seo/Sheen 1997, Ikehata 1998)

$$\int_{\partial\Omega} g(\Lambda(\sigma_1)-\Lambda(\sigma_2))g \, \mathrm{d}s \geq \int_\Omega (\sigma_2-\sigma_1)|\nabla u_{\sigma_2}^g|^2 \, \mathrm{d}x$$

for all $\sigma_1, \sigma_2 \in L^\infty_+(\Omega)$, $g \in L^2_\diamond(\partial\Omega)$.

- Monotonicity w.r.t. Loewner order:

$$\sigma_1 \leq \sigma_2 \quad \Longrightarrow \quad \Lambda(\sigma_1) \geq \Lambda(\sigma_2)$$

$\leadsto$ Inclusion detection method (Tamburrino/Rubinacci 2002)

- Localized potentials (**H.** 2008):

$$\exists (g_k)_{k\in\mathbb{N}}: \int_{D_1} |\nabla u_\sigma^{g_k}|^2 \, \mathrm{d}x \to \infty, \quad \int_{D_2} |\nabla u_\sigma^{g_k}|^2 \, \mathrm{d}x \to 0.$$

$\leadsto$ Converse monotonicity holds for inclusion detection.

$\leadsto$ Monotonicity method yields exact shape (**H.**/Ullrich 2013).

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Lemma.

$$\int_{\partial\Omega} g(\Lambda(\sigma_1)-\Lambda(\sigma_2))g \, \mathrm{d}s \geq \int_\Omega (\sigma_2-\sigma_1)|\nabla u_{\sigma_2}^g|^2 \, \mathrm{d}x$$
$$= \int_{\partial\Omega} g\Lambda'(\sigma_2)(\sigma_1-\sigma_2)g \, \mathrm{d}s.$$

for all $\sigma_1, \sigma_2 \in L_+^\infty(\Omega)$, $g \in L_\diamond^2(\partial\Omega)$.

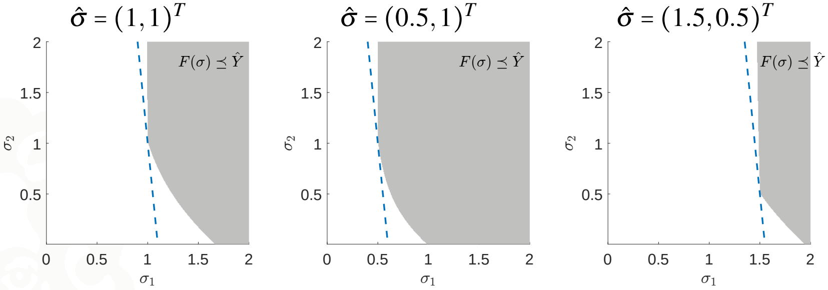
- For all $\sigma_1, \sigma_2 \in L_+^\infty(\Omega)$: $\Lambda(\sigma_1)-\Lambda(\sigma_2) \geq \Lambda'(\sigma_2)(\sigma_1-\sigma_2)$.
- Convexity: For all $\sigma_1, \sigma_2 \in L_+^\infty(\Omega)$, $t \in [0,1]$

$$\Lambda((1-t)\sigma_1 + t\sigma_2) \leq (1-t)\Lambda(\sigma_1) + t\Lambda(\sigma_2).$$

The "monotonicity lemma" also implies convexity.

Convexity for the simple example

Inverse problem: Reconstruct $\hat{\sigma} \in \mathbb{R}^2_+$ from $\hat{Y} = F(\hat{\sigma}) \in \mathbb{R}^{6\times 6}$

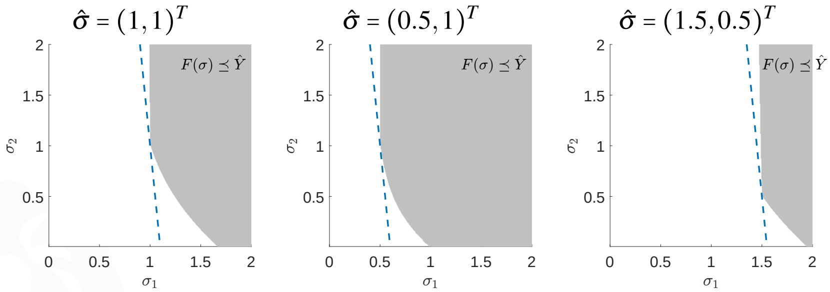


Observation.

$\hat{\sigma}$ is the lower left corner of the convex set $F(\sigma) \preceq \hat{Y}$.

("$\preceq$": Loewner / semidefiniteness order)

Mathematical formulation



Conjecture. $\exists c \in \mathbb{R}^n$ so that true solution $\hat{\sigma}$ minimizes

$$c^T \sigma = \sum_{i=1}^{n} c_i \sigma_i \to \text{min!} \quad \text{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \leq \hat{Y}.$$

For a similar but simpler Robin problem:

- Conjecture holds with $c = \mathbb{1}$ *(**H.**, Optim. Lett. 2021)*
- Global Newton convergence is possible *(**H.**, Numer. Math. 2021)*

Convex reformulation for EIT

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Theorem. (**H.**, *SIMA* 2023)

If sufficiently many measurements are taken, then:

- EIT forward mapping $F: [a,b]^n \to \mathbb{S}_m \subset \mathbb{R}^{m\times m}$ is injective.
- Derivative $F'(\sigma)$ is injective for all $\sigma \in [a,b]^n$.
- There exists $c \in \mathbb{R}^n_+$ so that for all $\hat{\sigma} \in [a,b]^n$, $\hat{Y} = \Lambda(\hat{\sigma})$:

  $\hat{\sigma}$ is the unique solution of the convex problem

$$\texttt{minimize} \quad c^T\sigma = \sum_{i=1}^n c_i\sigma_i \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \preceq \hat{Y}.$$

The Calderón problem with finitely many unknowns is equivalent to convex semidefinite optimization

Theorem (continued). (**H.**, *SIMA* 2023)
There exists $\lambda > 0$ so that

- for all $\hat{\sigma} \in [a,b]^n$, and $\hat{Y} := \Lambda(\hat{\sigma})$,
- and all $\delta > 0$, and $Y^\delta \in \mathbb{S}_m \subset \mathbb{R}^{m\times m}$, with $\|Y^\delta - \hat{Y}\| \leq \delta$,

the convex semidefinite optimization problem

$$\texttt{minimize} \quad c^T\sigma = \sum_{i=1}^n c_i\sigma_i \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \leq Y^\delta + \delta I.$$

possesses a minimizer σ^δ. Every such minimizer fulfills

$$\|\sigma^\delta - \hat{\sigma}\|_{c,\infty} \leq \frac{n-1}{\lambda}\delta.$$

($\|\cdot\|_{c,\infty}$: c-weighted maximum norm)

Error estimates for noisy data $Y^\delta \approx \hat{Y}$ also hold.

Non-constructive results vs. explicit estimates

The Calderón problem with finitely many unknowns

- is uniquely solvable for sufficiently many measurements
- is equivalent to some linear minimization over convex set
- allows some error estimate for noisy data

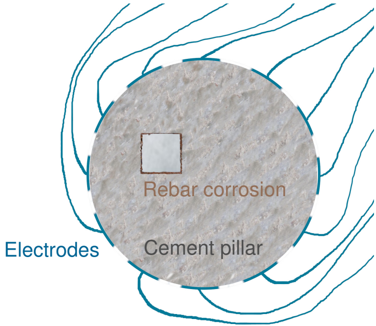
But (for a given setting and desired resolution)...

- How many measurements do we need for a given resolution?
- What linear functional should we minimize? ($c = ?$)
- What are the constants in the error estimate? ($\lambda = ?$)

Next slides: Explicit answers, but only for a Robin problem (similar but simpler than EIT)

An inverse Robin coefficient problem

EIT for corrosion detection



Non-destructive EIT-based corrosion detection:

- Apply electric currents on outer boundary $\partial\Omega$
- Measure necessary voltages

$\leadsto$ Detect corrosion on inner boundary $\Gamma = \partial D$

Idealized mathematical model: Robin PDE

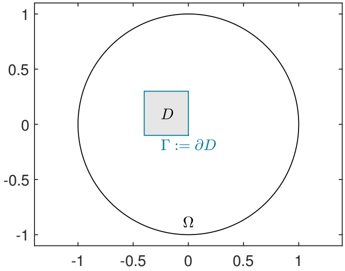
Electric potential $u:\ \Omega \to \mathbb{R}$ solves

(1) $\Delta u = 0$ in $\Omega \setminus \Gamma$,

(2) $\partial_\nu u|_{\partial\Omega} = g$ on $\partial\Omega$,

(3) $[\![u]\!]_\Gamma = 0$ on Γ,

(4) $[\![\partial_\nu u]\!]_\Gamma = \sigma u$ on Γ



Inverse Problem: Recover σ from Neumann-to-Dirichlet-Operator

$$\Lambda(\sigma):\ L^2(\partial\Omega) \to L^2(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves Robin PDE (1)–(4).

Finitely many measurements and unknowns

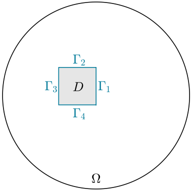
- Finitely many measurements:

$$\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, \mathrm{d}s \quad \text{for finitely many } g_1, \ldots, g_m$$

- Finite desired resolution:

$$\sigma = \sum_{j=1}^{n} \sigma_j \chi_{\Gamma_j} \quad \text{with } \sigma_j \in \mathbb{R},\ j = 1, \ldots, n$$

with partition $\Gamma = \bigcup_{j=1}^{n} \Gamma_j$



- A-priori bounds: $\sigma := (\sigma_1, \ldots, \sigma_n)^T \in [a,b]^n$, $b > a > 0$ known

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Finite-dimensional non-linear inverse problem: Determine

$$\sigma = (\sigma_j)_{j=1}^{n} \in [a,b]^n \quad \text{from} \quad F(\sigma) := \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, \mathrm{d}s \right)_{j,k=1}^{m} \in \mathbb{R}^{m\times m}$$

Explicit answers 1/3: *"what linear cost functional?"*

Theorem. (**H.**, Optim. Lett. 2021)

If sufficiently many measurements are taken, then

- $\hat{Y} := F(\hat{\sigma}) \in \mathbb{R}^{m\times m}$ uniquely determines $\hat{\sigma} \in [a,b]^n$.
- $\hat{\sigma}$ is the unique solution of

$$\texttt{minimize} \quad \|\sigma\|_1 = \sum_{j=1}^{n} \sigma_j \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \leq \hat{Y}.$$

- The constraint set $\sigma \in [a,b]^n,\ F(\sigma) \leq \hat{Y}$ is convex.

Explicitly known cost functional ($c = \mathbb{1}$)

Explicit answers 2/3: *"how many measurements?"*

Theorem. (**H.**, Optim. Lett. 2021)

- Suff. many measurements are taken if $\lambda_{\text{max}}(F'(z_{j,k})d_j) > 0$ for
$z_{j,k} := \frac{a}{2}e'_j + \left(a + k\frac{a}{4(n-1)}\right)e_j \in \mathbb{R}^n_+, \quad d_j := \frac{2b-a}{a}(n-1)e'_j - \frac{1}{2}e_j \in \mathbb{R}^n,$
with $j = 1, \ldots, n, \quad k = 1, \ldots, \lceil \frac{4(n-1)b}{a} \rceil - 4n + 5.$
- This criterion is fulfilled if $(g_j)_{j=1}^{\infty}$ has dense span in $L^2(\partial\Omega)$, and sufficiently many g_j are used.

($e_j \in \mathbb{R}^n$: j-th unit vector, $e'_j := \mathbb{1} - e_j \in \mathbb{R}^n$: negated j-th unit vector)

Explicitly computable criterion (finitely many PDE-solutions!)

Explicit answers 3/3: *"constants in error estimate?"*

Theorem. (**H.**, Optim. Lett. 2021)

- Let the criterion hold with lower bound $\lambda > 0$.
- Let $\delta > 0$, and $Y^\delta \in \mathbb{R}^{m\times m}$ be symmetric with $\|\hat{Y} - Y^\delta\|_2 \leq \delta$.

Then there exist solutions of

$$\texttt{minimize} \quad \|\sigma\|_1 = \sum_{j=1}^{n} \sigma_j \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \leq Y^\delta + \delta I.$$

and every such minimum σ^δ fulfills

$$\|\hat{\sigma} - \sigma^\delta\|_\infty \leq \frac{2\delta(n-1)}{\lambda}$$

Explicitly computable error estimate

Conclusions 1/2

For elliptic coefficient inverse problems

- least-squares residuum functionals may be highly non-convex
- local minima are usually useless

Possible remedy

- utilize monotonicity & convexity with respect to Loewner order

Equivalent convex reformulations are possible

- globally convergent solution algorithms are possible
- error estimates for noisy data are possible

Explicit (computable) resolution and error estimates

- are possible, at least for simple problems

Conclusions 2/2 *(now getting very subjective)*

Some future challenges in inverse problems in PDEs:

We need to progress and extend...

FROM: Uniqueness results for infinite-dimensional DtN/NtD
TO: Resolution attainable from finitely many measurements

FROM: Stability results
TO: Error estimates (with computable constants)

FROM: Local convergence and non-convex residuum functionals
TO: Global convergence and convex functionals

Loewner monotonicity & convexity can help with these challenges.