Monotonicity and Convexity in inverse coefficient problems

Bastian Harrach

http://numerical.solutions

Institute of Mathematics, Goethe University Frankfurt, Germany

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Motivation: 1D inverse problems

- Given $\hat{y} \in \mathbb{R}$, $a < b$, consider 1D inverse problem:

$$\text{Determine} \quad \hat{x} \in [a,b] \quad \text{from} \quad f(\hat{x}) = \hat{y},$$

  with given $\hat{y} \in \mathbb{R}$, $a < b$, and $f: [a,b] \to \mathbb{R}$ continuous.

- If f is strictly monotonically increasing:

$$\text{Inverse problem uniquely solvable} \quad \Longleftrightarrow \quad f(a) \leq \hat{y} \leq f(b).$$

  $\rightsquigarrow$ Easy-to-check criterion, requires only 2 function evals.

- If, additionally, f is convex

$$\text{Newton-Method converges for any } x^{(0)} \geq \hat{x}.$$

  $\rightsquigarrow$ Global Newton-convergence for $x^{(0)} := b$.

Monotonicity & convexity greatly helps in inverse problems.

Motivation: Calderón problem / EIT

NtD $\Lambda: L^\infty_+(\Omega) \to \mathcal{L}(L^2_\diamond(\partial\Omega))$ fulfills $\forall \sigma_1, \sigma_2 \in L^\infty_+(\Omega),\ g \in L^2_\diamond(\partial\Omega)$:

$$\begin{aligned}\int_{\partial\Omega} g(\Lambda(\sigma_1) - \Lambda(\sigma_2))g \,\mathrm{d}s &\geq \int_\Omega (\sigma_2 - \sigma_1)|\nabla u^g_{\sigma_2}|^2 \,\mathrm{d}x \\ &= \int_{\partial\Omega} g\Lambda'(\sigma_2)(\sigma_1 - \sigma_2)g \,\mathrm{d}s.\end{aligned}$$

- Monotonicity: $\sigma_1 \leq \sigma_2 \implies \Lambda(\sigma_1) \geq \Lambda(\sigma_2)$
- Convexity: $\Lambda(\sigma_1) - \Lambda(\sigma_2) \geq \Lambda'(\sigma_2)(\sigma_1 - \sigma_2)$

  w.r.t. Loewner order

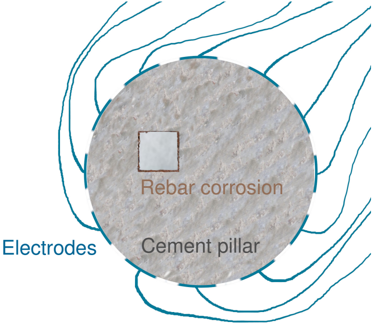
$$A \geq B \quad :\iff \quad B - A \text{ positive semidefinite.}$$

This talk: Utilizing monotonicity & convexity for Robin problem
(similar but simpler than Calderón problem/EIT)

An inverse Robin coefficient problem

(with applications in corrosion detection)

Electrical Impedance Tomography for corrosion detection



Non-destructive EIT-based corrosion detection:

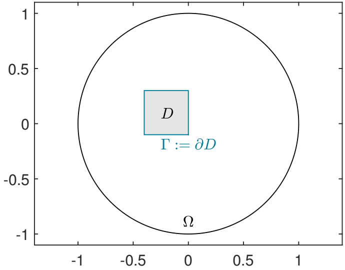
- Apply electric currents on outer boundary $\partial\Omega$
- Measure necessary voltages

$\rightsquigarrow$ Detect corrosion on inner boundary $\Gamma = \partial D$

Idealized mathematical model: Robin PDE

Electric potential $u: \ \Omega \to \mathbb{R}$ solves

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega \setminus \Gamma, \\ \partial_\nu u|_{\partial\Omega} &= g && \text{on } \partial\Omega, \\ [\![u]\!]_\Gamma &= 0 && \text{on } \Gamma, \\ [\![\partial_\nu u]\!]_\Gamma &= \gamma u && \text{on } \Gamma \end{aligned}$$



- *Applied boundary currents:* $g: \ \partial\Omega \to \mathbb{R}$
- *Corrosion coefficient:* $\gamma: \ \Gamma \to \mathbb{R}$
- *Voltage jump:* $[\![u]\!] := u^+|_\Gamma - u^-|_\Gamma$
- *Lack of electrical currents:* $[\![\partial_\nu u]\!] := \partial_\nu u^+|_\Gamma - \partial_\nu u^-|_\Gamma$
- *Measured boundary voltages:* $u|_{\partial\Omega}: \ \partial\Omega \to \mathbb{R}$

Global uniqueness from idealized data

Theorem. (**H.**/Meftahi, SIAM J. Appl. Math. 2019)
$\gamma \in L^\infty_+(\Gamma)$ is uniquely determined by *Neumann-Dirichlet-Operator*

$$\Lambda(\gamma):\ L^2(\partial\Omega) \to L^2(\partial\Omega), \quad g \mapsto u_\gamma^{(g)}|_{\partial\Omega},$$

where $u_\gamma^{(g)}$ solves Robin PDE (1)–(4).

$\rightsquigarrow$ Infinitely many measurements with infinite accuracy uniquely determine $\gamma \in L^\infty_+(\Gamma)$ with infinite resolution.

Consequences for practical applications?

Towards practical applications

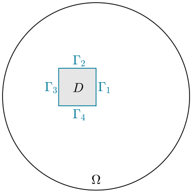
- Finitely many measurements:

$$\int_{\partial\Omega} g_j \Lambda(\gamma) g_j \, \mathrm{d}s \quad \text{for finitely many } g_j,\ j = 1, \ldots, m$$

(power required to keep up current g_j, electrode models yield similar expressions)

- Finite desired resolution:

$$\gamma = \sum_{j=1}^{n} \gamma_j \chi_{\Gamma_j} \quad \text{with } \gamma_j \in \mathbb{R},\ j = 1, \ldots, n$$



with partition $\Gamma = \bigcup_{j=1}^{n} \Gamma_j$

- A-priori bounds: $\gamma := (\gamma_1, \ldots, \gamma_n)^T \in [a, b]^n$ with known $b > a > 0$.

Finite-dimensional non-linear inverse problem: Determine

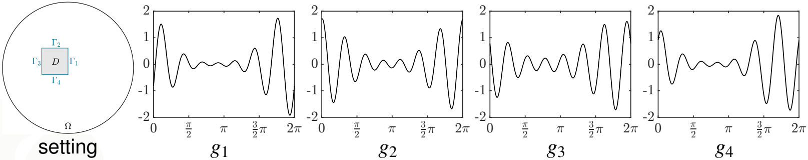
$$\gamma = (\gamma_j)_{j=1}^n \in [a,b]^n \quad \text{from} \quad F(\gamma) := \left(\int_{\partial\Omega} g_j \Lambda(\gamma) g_j \, \mathrm{d}s \right)_{j=1}^m \in \mathbb{R}^m$$

- Uniqueness: How many (and what) g_j make F injective?
- Stability/error estimates?
- How to determine γ from $F(\gamma)$? Convergence (local/global)?

Problem is much harder than the infinite-dimensional version!
But (for this simple Robin example): it can be solved.

Example result

Four unknown conductivities $\gamma_1,\ldots,\gamma_4$ with a-priori bounds $1 \leq \gamma_j \leq 2$

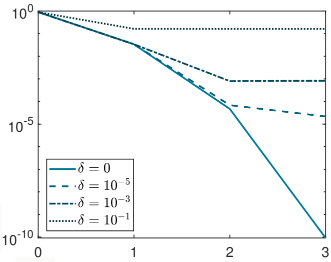


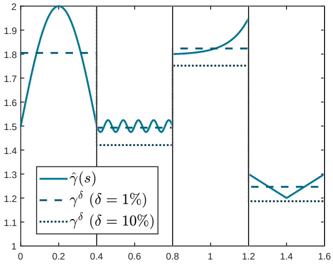
For $F(\gamma) := \left(\int_{\partial\Omega} g_j \Lambda(\gamma) g_j \, \mathrm{d}s\right)_{j=1}^4$ with these $g_1,\ldots,g_4$ we can prove

(**H.**, Numer. Math. 2020)

- $F(\gamma)$ uniquely determines $\gamma \in [1,2]^4$
- $\|\gamma-\gamma'\|_\infty \leq 7.5 \dfrac{\|F(\gamma)-F(\gamma')\|_\infty}{\|F(2)-F(1)\|_\infty}$ for all $\gamma,\gamma' \in [1,2]^4$
- Newton iteration with $\gamma^{(0)} = (1,1,1,1)$ (globally!) converges.

Noisy measurements





Using $F(\gamma) := \left(\int_{\partial\Omega} g_j \Lambda(\gamma) g_j \,\mathrm{d}s\right)_{j=1}^4$ with $g_1,\ldots,g_4$ as on last slide:

- Newton convergence speed is quadratic
- For all $y^\delta \in [F(2),F(1)]^4$ there exists unique γ with $F(\gamma) = y^\delta$
  - $\rightsquigarrow$ Lipschitz stability yields error estimate.
  - $\rightsquigarrow$ Newton finds pcw.-const. approx. if true γ is not pcw.-const.

Rest of talk: How to construct $g_1,\ldots,g_4$ and prove such results.

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Uniqueness, stability and global Newton convergence

(for pointwise convex monotonic functions)

Pointwise convex, monotonic C^1 functions

Given $F: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $m \geq n \geq 2$, on convex open set $U \subseteq \mathbb{R}^n$.

$$F \text{ pointwise monotonic} \iff F'(x) \geq 0 \qquad \forall x \in U,$$
$$F \text{ pointwise convex} \iff F(y) - F(x) \geq F'(x)(y-x) \quad \forall x, y \in U.$$

Goal: Find criteria that ensure

- Injectivity of F
- Lipschitz continuity of F^{-1}
- Global convergence of Newton's method for $n = m$

*Results known for **inverse** monotonic convex F, i.e. $F'(x)^{-1} \geq 0$.*

*We need results for **forward** monotonic convex F, i.e. $F'(x) \geq 0$.*

Simple version of the main result

Theorem. (**H.**, Numer. Math. 2020)
$F: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $F \in C^1$, pointwise convex and monotonic. If

$$U \supset [-1,3]^n \quad \text{and} \quad F'(-e_j + 3e_j')(e_j - 3e_j') \not\leq 0 \ \forall j = 1, \ldots, n,$$

then F is injective on $[0,1]^n$.

$e_j := (0 \ \ldots \ 0 \ 1 \ 0 \ \ldots \ 0)^T \in \mathbb{R}^n$ *unit vector,* $e_j' := 1 - e_j = (1 \ \ldots \ 1 \ 0 \ 1 \ \ldots \ 1)^T \in \mathbb{R}^n$

- Easy and simple-to-check criterion for injectivity
- Also yields injectivity of $F'(x)$ & Lipschitz continuity of F^{-1} with

$$L = 2\left(\min_{j=1,\ldots,n} \ \max_{k=1,\ldots,m} e_k^T F'(-e_j + 3e_j')\left(e_j - 3e_j'\right)\right)^{-1}$$

Global Newton convergence

Theorem. (**H.**, Numer. Math. 2020)
$F: U \subseteq \mathbb{R}^n \to \mathbb{R}^n$, $F \in C^1$, pointwise convex and monotonic. If

$$[-2, n(n+3)]^n \subset U \quad \text{and} \quad F'(z^{(j)})d^{(j)} \not\leq 0 \quad \text{for all } j \in \{1, \dots, n\},$$

with $z^{(j)} := -2e_j + n(n+3)e'_j$, and $d^{(j)} := e_j - (n^2 + 3n + 1)e'_j$, then

- F is injective on $[-1, n]^n$, $F'(x)$ is invertible for all $x \in [-1, n]^n$.
- If, additionally, $F(0) \leq 0 \leq F(1)$, then there exists a unique

  $$\hat{x} \in \left(-\tfrac{1}{n-1}, 1 + \tfrac{1}{n-1}\right)^n \quad \text{with} \quad F(\hat{x}) = 0,$$

  The Newton iteration started with $x^{(0)} := 1$ converges against $\hat{x}$.

Back to the Robin interface problem

- Monotonicity relations (Kang/Seo/Sheen 97, Ikehata 98, **H.**/Ullrich 13)

$$F(\gamma) := \left(\int_{\partial\Omega} g_j \Lambda(\gamma) g_j \, \mathrm{d}s \right)_{j=1}^{m} \in \mathbb{R}^m$$

  is pointw. convex and monot. decreasing for any choice of g_j.

- $F \in C^1$, directional derivatives fulfill, e.g.

$$F'(\gamma)(-e_j + 3e_j') = \left(\int_{\Gamma_j} |u_\gamma^{g_k}|^2 \, \mathrm{d}s - 3 \int_{\Gamma \setminus \Gamma_j} |u_\gamma^{g_k}|^2 \, \mathrm{d}s \right)_{k=1}^{n} \in \mathbb{R}^n$$

$\rightsquigarrow F'(z^{(j)})d^{(j)} \not\leq 0$ *if* $u_{z^{(j)}}^{g_j}$ *has high energy on* Γ_j *and low on* $\Gamma \setminus \Gamma_j$.

Back to the Robin interface problem

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$\rightsquigarrow F'(z^{(j)})d^{(j)} \not\leq 0$ *if* $u_{z^{(j)}}^{g_j}$ *has high energy on* Γ_j *and low on* $\Gamma \setminus \Gamma_j$.

- Localized potentials (**H.** 08): g_j can be chosen so that

$$F'(z^{(j)})d^{(j)} \not\leq 0 \quad \forall j$$

- Simultaneously localized potentials: g_j can be chosen so that

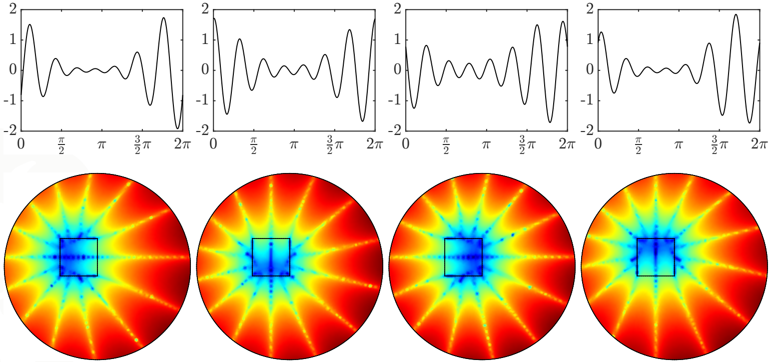
$$F'(z^{(j,k)})d^{(j)} \not\leq 0 \quad \forall j,k$$

(**H.**/Lin 20, important for treating $\gamma \in [a,b]^n$ with arbitrary $b > a > 0$)

- "g_j can be chosen": every large enough fin.-dim. subspace of $L^2(\partial\Omega)$ contains such g_j & explicit method to calculate them.

Back to the Robin interface problem

For the example result with four unknown conductivities $\gamma \in [1,2]^4$



$\log|\nabla u_\gamma^{g_1}|^2$ $\log|\nabla u_\gamma^{g_2}|^2$ $\log|\nabla u_\gamma^{g_3}|^2$ $\log|\nabla u_\gamma^{g_4}|^2$

- $u_\gamma^{g_j}$ has localized energy on Γ_j for certain γ

  (More precisely: for $K = 173$ choices of γ)

Conclusions and Outlook (1/2)

For fin.-dim. inverse problems with convex monotonic functions

- simple criterion ensures uniqueness and Lipschitz stability
- also yields global Newton convergence
- criterion requires to check finitely many directional derivatives

For a discretized inverse Robin coefficient problem

- assumptions connected to monotonicity & localized potentials
- boundary currents can be found that uniquely and stably determine conductivity with global Newton convergence

Extension: Reformulate problem as convex semidefinite program

- Robin Problem: **H.**, Optim. Lett., 2022
- Calderón problem: **H.**, SIMA, 2023

Conclusions and Outlook (2/2)

- Provocative claim:

  Finite-dimensional inverse coefficients problem are much harder than infinite-dimensional ones.

- Relation to classical Collatz theory:

  Elliptic PDE forward problems lead to inverse monotonic convex functions. Inverse elliptic coefficient problems lead to forward monotonic convex functions.