

# Inverse Problems in Partial Differential Equations

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# Introduction to inverse problems

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Pierre Simon Laplace (1814):

*"An intellect which ... would know  
all forces ... and all positions of all items,  
if this intellect were also vast enough to  
submit these data to analysis ...*

*for such an intellect nothing would be  
uncertain and the future just like the past  
would be present before its eyes."*



## Computational Science:

*If we know all necessary parameters, then we can numerically predict the outcome of an experiment (by mathematical formulas).*

### Goals:

- ▶ Prediction
- ▶ Optimization
- ▶ Inversion/Identification

## Computational Science

Generic simulation problem:

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Given input  $x$  calculate outcome  $y = F(x)$ .

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$x \in X$ : parameters / input

$y \in Y$ : outcome / measurements

$F : X \rightarrow Y$ : functional relation / model

Goals:

- ▶ **Prediction:** Given  $x$ , calculate  $y = F(x)$ .
- ▶ **Optimization:** Find  $x$ , such that  $F(x)$  is optimal.
- ▶ **Inversion/Identification:** Given  $F(x)$ , calculate  $x$ .

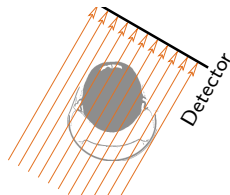
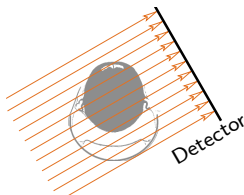
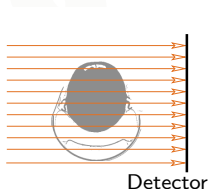
## Example: X-ray computerized tomography (CT)

Nobel Prize in Physiology or Medicine 1979:  
Allan M. Cormack and Godfrey N. Hounsfield

(Photos: Copyright ©The Nobel Foundation)



**Idea:** Take x-ray images from several directions



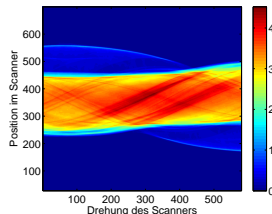
# Computerized tomography (CT)

(Image: Hanke-Bourgeois, Grundlagen der Numerischen Mathematik und des Wiss. Rechnens, Teubner 2002)



Image

$F$



Measurements

Direct problem:

Simulate/predict the measurements

(from knowledge of the interior density distribution)

*Given  $x$  calculate  $F(x) = y!$*

Inverse problem:

Reconstruct/image the interior distribution

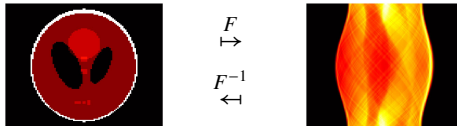
(from taking x-ray measurements)

*Given  $y$  solve  $F(x) = y!$*

# Computerized tomography

- ▶ CT forward operator  $F : x \mapsto y$  is linear
- ↪ Evaluation of  $F$  is simple matrix vector multiplication  
(after discretizing image and measurements as long vectors)

Simple low resolution example:




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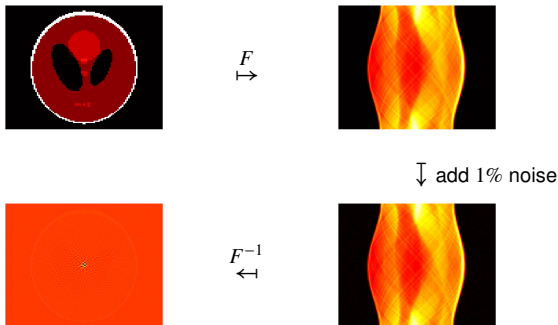
Problem: Matrix  $F$  invertible, but  $\|F^{-1}\|$  very large.

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## Ill-posedness

- ▶ In the continuous case:  $F^{-1}$  not continuous
- ▶ After discretization:  $\|F^{-1}\|$  very large




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*Are stable reconstructions impossible?*

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## Ill-posedness

### Generic linear ill-posed inverse problem

- ▶  $F : X \rightarrow Y$  bounded and linear,  $X, Y$  Hilbert spaces,
- ▶  $F$  injective,  $F^{-1}$  not continuous,
- ▶ True solution and noise-free measurements:  $F\hat{x} = \hat{y}$ ,
- ▶ Real measurements:  $y^\delta$  with  $\|y^\delta - \hat{y}\| \leq \delta$

$$F^{-1}y^\delta \not\rightarrow F^{-1}\hat{y} = \hat{x} \quad \text{for} \quad \delta \rightarrow 0.$$

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Even the smallest noise may corrupt the reconstructions.

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# Regularization

## Generic linear Tikhonov regularization

$$R_{\alpha} = (F^*F + \alpha I)^{-1}F^*$$

$\leadsto R_{\alpha}$  continuous,  $x = R_{\alpha}y^{\delta}$  minimizes

$$\|Fx - y^{\delta}\|^2 + \alpha \|x\|^2 \rightarrow \min!$$

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**Theorem.** Choose  $\alpha := \delta$ . Then for  $\delta \rightarrow 0$ ,

$$R_{\delta}y^{\delta} \rightarrow F^{-1}\hat{y}.$$


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## Regularization

**Theorem.** Choose  $\alpha := \delta$ . Then for  $\delta \rightarrow 0$ ,

$$R_\delta y^\delta \rightarrow F^{-1} \hat{y}.$$

**Proof.** Show that  $\|R_\alpha\| \leq \frac{1}{\sqrt{\alpha}}$  and apply

$$\|R_\alpha y^\delta - F^{-1} \hat{y}\| \leq \underbrace{\|R_\alpha (y^\delta - \hat{y})\|}_{\leq \|R_\alpha\| \delta} + \underbrace{\|R_\alpha \hat{y} - F^{-1} \hat{y}\|}_{\rightarrow 0 \text{ for } \alpha \rightarrow 0}.$$

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Inexact but continuous reconstruction (**regularization**)

+ Information on measurement noise (**parameter choice rule**)

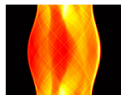
= Convergence

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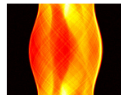
# Example ( $\delta = 1\%$ )



$\hat{x}$



$\hat{y} = F \hat{x}$



$y^\delta$



$F^{-1}y^\delta$



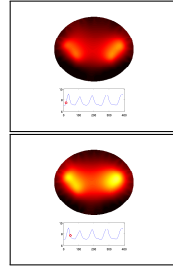
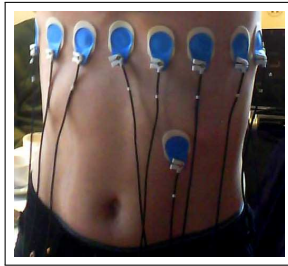
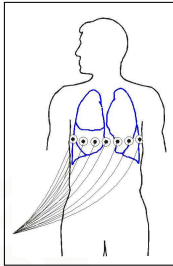
$(F^*F + \delta I)^{-1}F^*y^\delta$

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# Inverse Problems in Partial Differential Equations

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# Electrical impedance tomography (EIT)

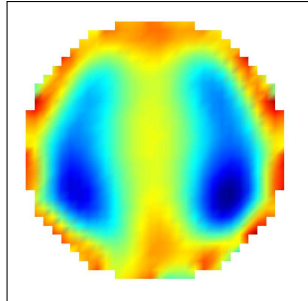
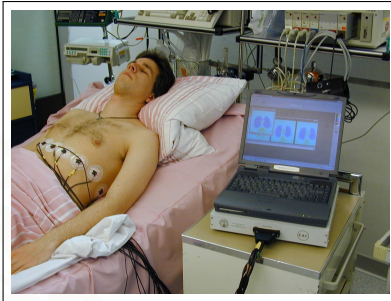


- ▶ Apply electric currents on subject's boundary
- ▶ Measure necessary voltages
- Reconstruct conductivity inside subject.

Images from BMBF-project on EIT

(Hanke, Kirsch, Kress, Hahn, Weller, Schilcher, 2007-2010)

## MF-System Goe-MF II



Electric current strength:  $5 - 500\text{mA}_{\text{rms}}$ , 44 images/second,  
CE certified by Viasys Healthcare, approved for clinical research



## Mathematical Model

- ▶ Electrical potential  $u(x)$  solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega \quad (\text{EIT})$$

$\Omega \subset \mathbb{R}^n$ : imaged body,  $n \geq 2$

$\sigma(x)$ : conductivity

$u(x)$ : electrical potential

- ▶ Idealistic model for boundary meas. (continuum model):

$\sigma \partial_\nu u(x)|_{\partial\Omega}$ : applied electric current

$u(x)|_{\partial\Omega}$ : measured boundary voltage (potential)

- ▶ Neumann-to-Dirichlet-Operator:

$$\Lambda(\sigma) : L^2_\diamond(\partial\Omega) \rightarrow L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

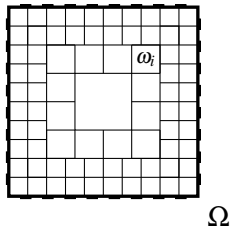
where  $u$  solves (EIT) with  $\sigma \partial_\nu u|_{\partial\Omega} = g$ .

## EIT in practice

- ▶ Finitely many unknowns,  $\sigma$  pcw. const. on given resolution  $\Omega = \bigcup_{i=1}^n \omega_i$
- ▶ Finitely many measurements

$$\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, ds$$

for given currents  $g_1, \dots, g_m \in L^2_{\diamond}(\partial\Omega)$



~> Finite-dimensional nonlinear inverse problem: Determine

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix} \in \mathbb{R}_+^n \quad \text{from } F(\sigma) = \left( \int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, ds \right)_{j,k=1}^m \in \mathbb{R}^{m \times m}.$$

## Mathematical challenges for practical EIT

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**Inverse problem:** Determine  $\sigma \in \mathbb{R}_+^n$  from  $Y = F(\sigma) \in \mathbb{R}^{m \times m}$ .

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For a fixed desired resolution:

- ▶ How many measurements uniquely determine  $\sigma$ ?
- ▶ Stability / error estimates for noisy data  $Y^\delta \approx F(\sigma)$ ?
- ▶ Numerical algorithm to determine  $\sigma \in \mathbb{R}_+^n$  from  $Y^\delta \approx F(\sigma)$ ?
- ▶ Global/local convergence of algorithm?

*Next slides: The problem of local convergence / local minimizers*

## Simple example: EIT with 2 unknowns & 6 bndry. currents

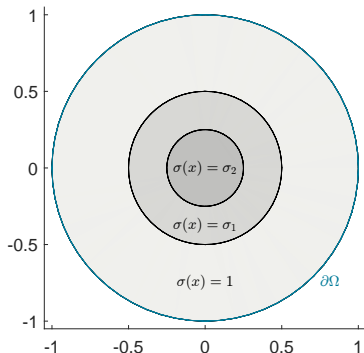
$\Omega$ : unit circle

$$F: \mathbb{R}_+^2 \rightarrow \mathbb{R}^{6 \times 6}$$

$$F \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} := \left( \int_{\partial\Omega} g_j \Lambda(\sigma) g_k \right)_{j,k=1}^6$$

with trigonometric currents

$$\{g_1, \dots, g_6\} = \{\sin(\varphi), \dots, \cos(3\varphi)\}$$

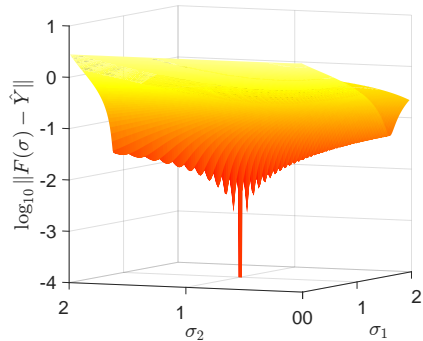
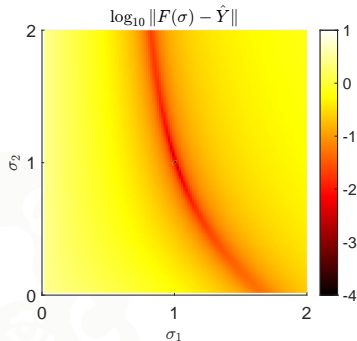


**Inverse problem:** Reconstruct  $\hat{\sigma} \in \mathbb{R}_+^2$  from  $\hat{Y} = F(\hat{\sigma}) \in \mathbb{R}^{6 \times 6}$

**Natural approach:** Least squares data fitting

$$\text{minimize} \quad \|F(\sigma) - \hat{Y}\|_F^2 \quad (+ \text{Regularization})$$

## Problem of local minima



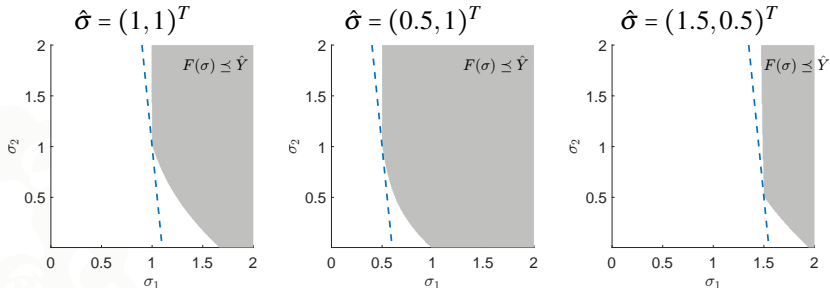
Numerical results indicate

- ▶  $\hat{Y} = F(\hat{\sigma})$  uniquely determines  $\hat{\sigma} \dots$
- ▶  $\dots$  but residuum is highly non-convex, many local minima

Are globally convergent algorithms impossible?

## Towards a convex reformulation

**Inverse problem:** Reconstruct  $\hat{\sigma} \in \mathbb{R}_+^2$  from  $\hat{Y} = F(\hat{\sigma}) \in \mathbb{R}^{6 \times 6}$

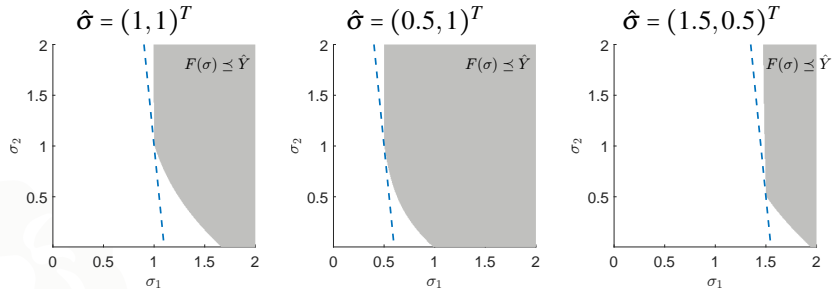


**Conjecture.**

$\hat{\sigma}$  is the lower left corner of the convex set  $F(\sigma) \leq \hat{Y}$ .

("≤": Loewner / semidefiniteness order)

## Towards a convex reformulation



**Conjecture.**  $\exists c \in \mathbb{R}^n$  so that true solution  $\hat{\sigma}$  minimizes

$$c^T \sigma = \sum_{i=1}^n c_i \sigma_i \rightarrow \min! \quad \text{s.t.} \quad \sigma \in [a, b]^n, F(\sigma) \leq \hat{Y}.$$

For a similar but simpler Robin problem:

- ▶ Conjecture holds with  $c = \mathbb{1}$  (*H., Optim. Lett. 2021*)
- ▶ Global Newton convergence is possible (*H., Numer. Math. 2021*)

## Convex reformulation for EIT

**Theorem.** (*H., SIAM J. Math. Anal., accepted*)

If sufficiently many measurements are taken, then:

- ▶ EIT forward mapping  $F : [a, b]^n \rightarrow \mathbb{S}_m \subset \mathbb{R}^{m \times m}$  is injective.
- ▶ Derivative  $F'(\sigma)$  is injective for all  $\sigma \in [a, b]^n$ .
- ▶ There exists  $c \in \mathbb{R}_+^n$  so that for all  $\hat{\sigma} \in [a, b]^n$ ,  $\hat{Y} = \Lambda(\hat{\sigma})$ :

$\hat{\sigma}$  is the unique solution of the convex problem

$$\text{minimize } c^T \sigma = \sum_{i=1}^n c_i \sigma_i \quad \text{s.t.} \quad \sigma \in [a, b]^n, F(\sigma) \leq \hat{Y}.$$

*The EIT (aka Calderón) problem with finitely many unknowns is equivalent to convex semidefinite optimization*



## Stability and error estimates

**Theorem (continued).** (*H., SIAM J. Math. Anal., accepted*)

There exists  $\lambda > 0$  so that

- ▶ for all  $\hat{\sigma} \in [a, b]^n$ , and  $\hat{Y} := \Lambda(\hat{\sigma})$ ,
- ▶ and all  $\delta > 0$ , and  $Y^\delta \in \mathbb{S}_m \subset \mathbb{R}^{m \times m}$ , with  $\|Y^\delta - \hat{Y}\| \leq \delta$ ,

the convex semidefinite optimization problem

$$\text{minimize } c^T \sigma = \sum_{i=1}^n c_i \sigma_i \quad \text{s.t.} \quad \sigma \in [a, b]^n, F(\sigma) \leq Y^\delta + \delta I.$$

possesses a minimizer  $\sigma^\delta$ . Every such minimizer fulfills

$$\|\sigma^\delta - \hat{\sigma}\|_{c, \infty} \leq \frac{n-1}{\lambda} \delta.$$

( $\|\cdot\|_{c, \infty}$ : *c-weighted maximum norm*)

*Error estimates for noisy data  $Y^\delta \approx \hat{Y}$  also hold.*

## Conclusions

### For inverse problems in elliptic PDEs

- ▶ least-squares residuum functionals may be highly non-convex
- ▶ local minima are usually useless

### Possible remedy

- ▶ utilize monotonicity & convexity with respect to Loewner order
- ▶ utilize localized potentials to control directional derivatives

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~> Equivalent convex reformulations are possible

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### Possible extensions: Monotonicity & localized potentials known for

- ▶ Helmholtz (H./Pohjola/Salo 19), Inverse Scattering (H./Griesmaier 18), Fractional Schrödinger (H./Lin 19), Robin Transmission (H./Meftahi 19), Elasticity (H./Eberle 22), Fractional Semilinear (Lin 22), ...