Inverse Problems in Partial Differential Equations

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Introduction to inverse problems

Pierre Simon Laplace (1814):

"An intellect which ... would know all forces ... and all positions of all items,

if this intellect were also vast enough to submit these data to analysis ...

for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes."

Computational Science:

If we know all necessary parameters, then we can numerically predict the outcome of an experiment (by mathematical formulas).

Goals:

- Prediction
- Optimization
- Inversion/Identification

Generic simulation problem:

Given input x calculate outcome $y = F(x)$.

$x \in X$:	parameters / input
$y \in Y$:	outcome / measurements
$F : X \to Y$:	functional relation / model

Goals:

- Prediction: Given x, calculate $y = F(x)$.
- Optimization: Find x, such that $F(x)$ is optimal.
- Inversion/Identification: Given $F(x)$, calculate x.

Example: X-ray computerized tomography (CT)

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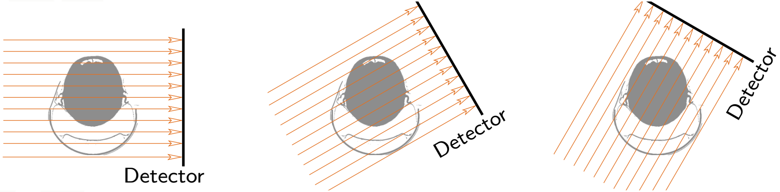
Nobel Prize in Physiology or Medicine 1979:
Allan M. Cormack and Godfrey N. Hounsfield

(Photos: Copyright ©The Nobel Foundation)

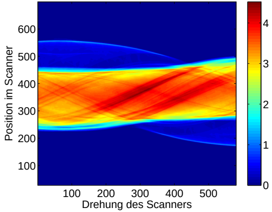
Idea: Take x-ray images from several directions



Computerized tomography (CT)

(Image: Hanke-Bourgeois, Grundlagen der Numerischen Mathematik und des Wiss. Rechnens, Teubner 2002)

$\overset{F}{\longmapsto}$



Image

Measurements

Direct problem: Simulate/predict the measurements

(from knowledge of the interior density distribution)

Given x *calculate* $F(x) = y$*!*

Inverse problem: Reconstruct/image the interior distribution

(from taking x-ray measurements)

Given y *solve* $F(x) = y$*!*

Computerized tomography

- CT forward operator $F: \ x \mapsto y$ is linear

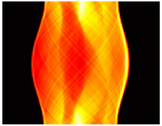
⇝ Evaluation of F is simple matrix vector multiplication

(after discretizing image and measurements as long vectors)

Simple low resolution example:

$\overset{F}{\mapsto}$

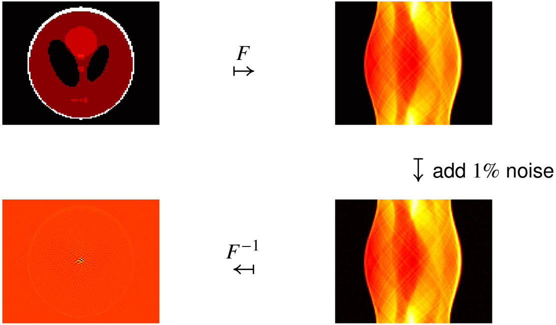
$\overset{F^{-1}}{\leftarrow\!\shortmid}$



Problem: Matrix F invertible, but $\|F^{-1}\|$ very large.

Ill-posedness

- In the continuous case: F^{-1} not continuous
- After discretization: $\|F^{-1}\|$ very large



Are stable reconstructions impossible?

Ill-posedness

Generic linear ill-posed inverse problem

- $F: X \to Y$ bounded and linear, X, Y Hilbert spaces,
- F injective, F^{-1} not continuous,
- True solution and noise-free measurements: $F\hat{x} = \hat{y}$,
- Real measurements: y^δ with $\|y^\delta - \hat{y}\| \leq \delta$

$$F^{-1}y^\delta \not\to F^{-1}\hat{y} = \hat{x} \quad \text{for} \quad \delta \to 0.$$

Even the smallest noise may corrupt the reconstructions.

Regularization

Generic linear Tikhonov regularization

$$R_\alpha = (F^*F + \alpha I)^{-1} F^*$$

$\rightsquigarrow R_\alpha$ continuous, $x = R_\alpha y^\delta$ minimizes

$$\|Fx - y^\delta\|^2 + \alpha \|x\|^2 \to \text{min!}$$

Theorem. Choose $\alpha := \delta$. Then for $\delta \to 0$,

$$R_\delta y^\delta \to F^{-1}\hat{y}.$$

Regularization

Theorem. Choose $\alpha := \delta$. Then for $\delta \to 0$,

$$R_\delta y^\delta \to F^{-1}\hat{y}.$$

Proof. Show that $\|R_\alpha\| \leq \frac{1}{\sqrt{\alpha}}$ and apply

$$\|R_\alpha y^\delta - F^{-1}\hat{y}\| \leq \underbrace{\|R_\alpha(y^\delta - \hat{y})\|}_{\leq \|R_\alpha\|\,\delta} + \underbrace{\|R_\alpha \hat{y} - F^{-1} y\|}_{\to 0 \text{ for } \alpha \to 0}.$$

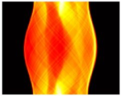
Inexact but continuous reconstruction (regularization)

\+ Information on measurement noise (parameter choice rule)

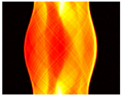
= Convergence

Example ($\delta = 1\%$)

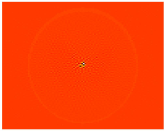
$\hat{x}$



$\hat{y} = F\hat{x}$



y^δ



$F^{-1} y^\delta$

$(F^* F + \delta I)^{-1} F^* y^\delta$

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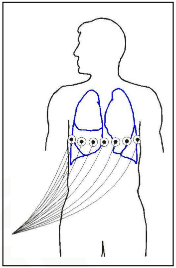
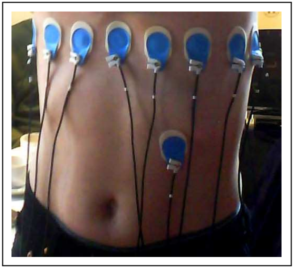
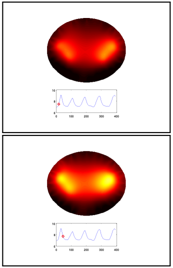
Inverse Problems in Partial Differential Equations

Electrical impedance tomography (EIT)







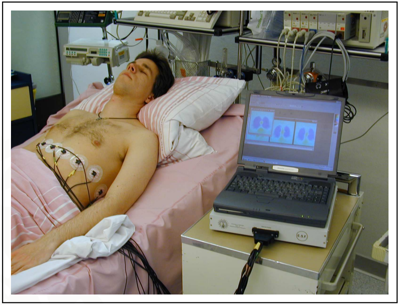
- Apply electric currents on subject's boundary
- Measure necessary voltages

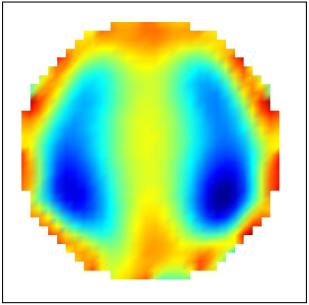
⇝ Reconstruct conductivity inside subject.

Images from BMBF-project on EIT

(Hanke, Kirsch, Kress, Hahn, Weller, Schilcher, 2007-2010)

MF-System Goe-MF II





Electric current strength: $5 - 500\mathrm{mA}_{\mathrm{rms}}$, 44 images/second,
CE certified by Viasys Healthcare, approved for clinical research

Mathematical Model

- Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega \qquad \text{(EIT)}$$

$\Omega \subset \mathbb{R}^n$:	imaged body, $n \geq 2$
$\sigma(x)$:	conductivity
$u(x)$:	electrical potential

- Idealistic model for boundary meas. (continuum model):

$\sigma \partial_\nu u(x)\|_{\partial\Omega}$:	applied electric current
$u(x)\|_{\partial\Omega}$:	measured boundary voltage (potential)

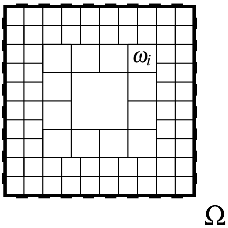
- Neumann-to-Dirichlet-Operator:

$$\Lambda(\sigma): L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (EIT) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

EIT in practice

- Finitely many unknowns, σ pcw. const. on given resolution $\Omega = \bigcup_{i=1}^{n} \omega_i$
- Finitely many measurements
$$\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, \mathrm{d}s$$
for given currents $g_1, \ldots, g_m \in L^2_\diamond(\partial\Omega)$



$\leadsto$ Finite-dimensional nonlinear inverse problem: Determine

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix} \in \mathbb{R}^n_+ \quad \text{from } F(\sigma) = \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, \mathrm{d}s \right)_{j,k=1}^{m} \in \mathbb{R}^{m\times m}.$$

Mathematical challenges for practical EIT

Inverse problem: Determine $\sigma \in \mathbb{R}^n_+$ from $Y = F(\sigma) \in \mathbb{R}^{m\times m}$.

For a fixed desired resolution:

- How many measurements uniquely determine σ?
- Stability / error estimates for noisy data $Y^\delta \approx F(\sigma)$?
- Numerical algorithm to determine $\sigma \in \mathbb{R}^n_+$ from $Y^\delta \approx F(\sigma)$?
- Global/local convergence of algorithm?

Next slides: The problem of local convergence / local minimizers

Simple example: EIT with 2 unknowns & 6 bndry. currents

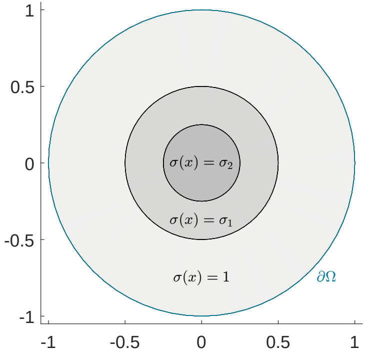
Ω: unit circle

$$F: \mathbb{R}_+^2 \to \mathbb{R}^{6\times 6}$$

$$F\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} := \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k\right)_{j,k=1}^6$$

with trigonometric currents

$$\{g_1,\ldots,g_6\} = \{\sin(\varphi),\ldots,\cos(3\varphi)\}$$

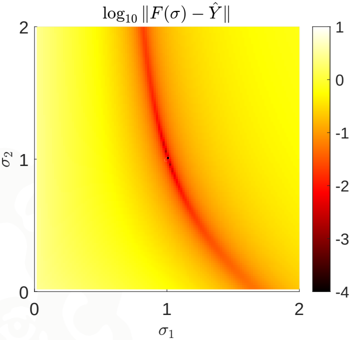


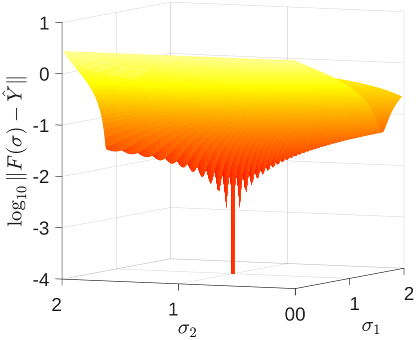
Inverse problem: Reconstruct $\hat{\sigma} \in \mathbb{R}_+^2$ from $\hat{Y} = F(\hat{\sigma}) \in \mathbb{R}^{6\times 6}$

Natural approach: Least squares data fitting

$$\texttt{minimize} \quad \|F(\sigma) - \hat{Y}\|_F^2 \qquad \textit{(+ Regularization)}$$

Problem of local minima





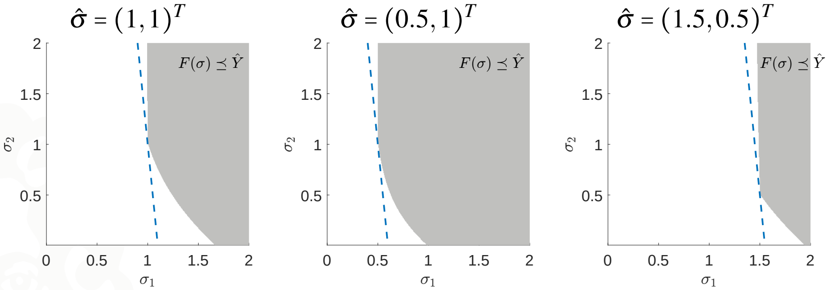
Numerical results indicate

- $\hat{Y} = F(\hat{\sigma})$ uniquely determines $\hat{\sigma}$...
- ...but residuum is highly non-convex, many local minima

Are globally convergent algorithms impossible?

Towards a convex reformulation

Inverse problem: Reconstruct $\hat{\sigma} \in \mathbb{R}^2_+$ from $\hat{Y} = F(\hat{\sigma}) \in \mathbb{R}^{6\times 6}$

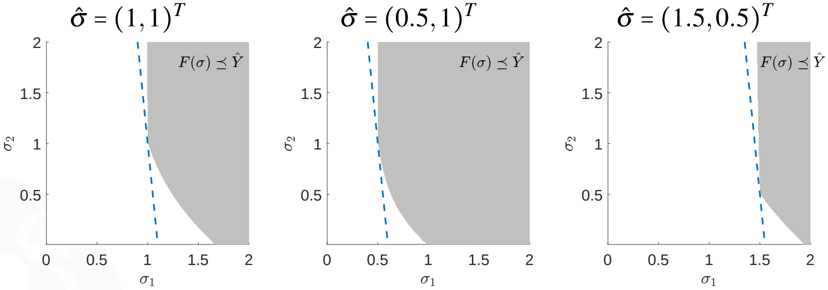


Conjecture.

$\hat{\sigma}$ is the lower left corner of the convex set $F(\sigma) \preceq \hat{Y}$.

("$\preceq$": Loewner / semidefiniteness order)

Towards a convex reformulation



Conjecture. $\exists c \in \mathbb{R}^n$ so that true solution $\hat{\sigma}$ minimizes

$$c^T \sigma = \sum_{i=1}^{n} c_i \sigma_i \to \text{min!} \quad \text{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \leq \hat{Y}.$$

For a similar but simpler Robin problem ($\rightsquigarrow$ Talk of Andrej Brojatsch):

- Conjecture holds with $c = \mathbb{1}$ *(**H.**, Optim. Lett. 2021)*
- Global Newton convergence is possible *(**H.**, Numer. Math. 2021)*

Convex reformulation for EIT

Theorem. (**H.**, arxiv:2203.16779)

If sufficiently many measurements are taken, then:

- EIT forward mapping $F: [a,b]^n \to \mathbb{S}_m \subset \mathbb{R}^{m\times m}$ is injective.
- Derivative $F'(\sigma)$ is injective for all $\sigma \in [a,b]^n$.
- There exists $c \in \mathbb{R}^n_+$ so that for all $\hat{\sigma} \in [a,b]^n$, $\hat{Y} = \Lambda(\hat{\sigma})$:

  $\hat{\sigma}$ is the unique solution of the convex problem

$$\texttt{minimize} \quad c^T\sigma = \sum_{i=1}^{n} c_i\sigma_i \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \preceq \hat{Y}.$$

The EIT (aka Calderón) problem with finitely many unknowns is equivalent to convex semidefinite optimization

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Theorem (continued). (**H.**, arxiv:2203.16779)

There exists $\lambda > 0$ so that

- for all $\hat{\sigma} \in [a,b]^n$, and $\hat{Y} := \Lambda(\hat{\sigma})$,
- and all $\delta > 0$, and $Y^\delta \in \mathbb{S}_m \subset \mathbb{R}^{m \times m}$, with $\|Y^\delta - \hat{Y}\| \leq \delta$,

the convex semidefinite optimization problem

$$\texttt{minimize} \quad c^T\sigma = \sum_{i=1}^n c_i\sigma_i \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \leq Y^\delta + \delta I.$$

possesses a minimizer σ^δ. Every such minimizer fulfills

$$\|\sigma^\delta - \hat{\sigma}\|_{c,\infty} \leq \frac{n-1}{\lambda}\delta.$$

($\|\cdot\|_{c,\infty}$: c-weighted maximum norm)

Error estimates for noisy data $Y^\delta \approx \hat{Y}$ also hold.

Conclusions

For inverse problems in elliptic PDEs

- least-squares residuum functionals may be highly non-convex
- local minima are usually useless

Possible remedy

- utilize monotonicity & convexity with respect to Loewner order
- utilize localized potentials to control directional derivatives

Equivalent convex reformulations are possible

- globally convergent solution algorithms are possible
- error estimates for noisy data are possible
- For similar but simpler Robin problem ($\rightsquigarrow$ Talk of A. Brojatsch)
  - explicit characterizations of achievable resolution
  - explicit error estimates for noisy data