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From inverse coefficient problems to convex semidefinite optimization

Bastian Harrach

http://numerical.solutions

Institute of Mathematics, Goethe University Frankfurt, Germany

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Electrical impedance tomography (EIT)

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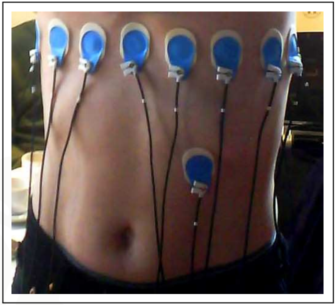
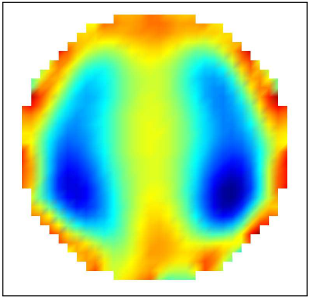




- Apply electric currents on subject's boundary
- Measure necessary voltages

⇝ Reconstruct conductivity inside subject

Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \subset \mathbb{R}^d \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves (1)}\}\,?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma) : \; L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Challenges in idealized EIT

Mathematical idealization of EIT $\rightsquigarrow$ Calderón problem

- infinitely many unknowns $\sigma \in L_+^\infty(\Omega)$
- infinitely many measurements $\Lambda(\sigma) \in \mathcal{L}(L_\diamond^2(\partial\Omega))$
- nonlinear forward map $\sigma \mapsto \Lambda(\sigma)$

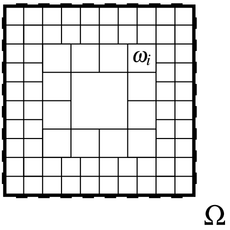
Mathematical challenges

- Uniqueness? Does $\Lambda(\sigma)$ determine σ?
- Stability? $\Lambda^{-1}: \Lambda(\sigma) \mapsto \sigma$ continuous?
- Convergence (local/global)? How to determine σ from $\Lambda(\sigma)$?

Consequences for practical EIT?

EIT in practice

- Finitely many unknowns, σ pcw. const. on given resolution $\Omega = \bigcup_{i=1}^n \omega_i$
- Finitely many measurements
$$\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, \mathrm{d}s$$
for given currents $g_1, \ldots, g_m \in L^2_\diamond(\partial\Omega)$



Finite-dimensional inverse problem: Determine

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix} \in \mathbb{R}^n_+ \quad \text{from } F(\sigma) = \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, \mathrm{d}s \right)_{j,k=1}^m \in \mathbb{R}^{m\times m}.$$

Mathematical challenges for practical EIT

Inverse problem: Determine $\sigma \in \mathbb{R}_+^n$ from $Y = F(\sigma) \in \mathbb{R}^{m\times m}$.

For a fixed desired resolution:

- How many measurements uniquely determine σ?
- Stability / error estimates for noisy data $Y^\delta \approx F(\sigma)$?
- Numerical algorithm to determine $\sigma \in \mathbb{R}_+^n$ from $Y^\delta \approx F(\sigma)$?
- Global/local convergence of algorithm?

Next slides: The problem of local convergence

Simple example: EIT with 2 unknowns & 6 bndry. currents

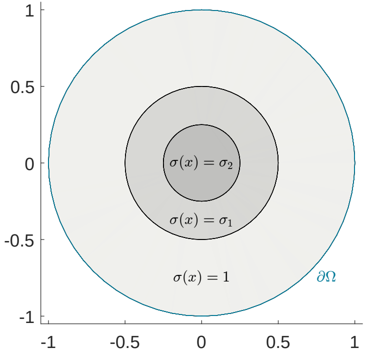
Ω: unit circle

$$F:\ \mathbb{R}^2_+ \to \mathbb{R}^{6\times 6}$$

$$F\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} := \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k\right)_{j,k=1}^{6}$$

with trigonometric currents

$$\{g_1,\ldots,g_6\} = \{\sin(\varphi),\ldots,\cos(3\varphi)\}$$

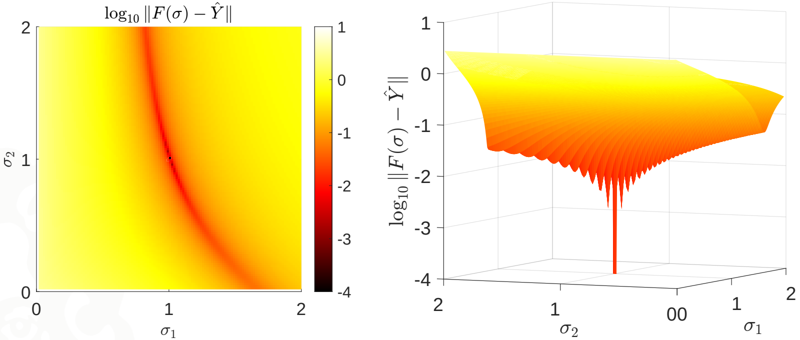


Inverse problem: Reconstruct $\hat{\sigma} \in \mathbb{R}^2_+$ from $\hat{Y} = F(\hat{\sigma}) \in \mathbb{R}^{6\times 6}$

Natural approach: Least squares data fitting

$$\texttt{minimize} \quad \|F(\sigma) - \hat{Y}\|_F^2 \qquad \textit{(+ Regularization)}$$

Problem of local minima

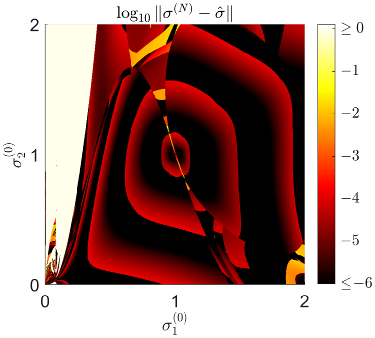


Numerical results indicate

- $\hat{Y} = F(\hat{\sigma})$ uniquely determines $\hat{\sigma}\ldots$
- ...but residuum is highly non-convex, many local minima

Impossible to find global minimizer?

Problem of local convergence



- Error of Matlab's `lsqnonlin` depends on initial values

Are globally convergent algorithms impossible?

Monotonicity and Convexity

The Monotonicity Lemma

Lemma. (First appearance: Kang/Seo/Sheen 1997, Ikehata 1998)

$$\int_{\partial\Omega} g(\Lambda(\sigma_1) - \Lambda(\sigma_2))g \, \mathrm{d}s \geq \int_\Omega (\sigma_2 - \sigma_1)|\nabla u_{\sigma_2}^g|^2 \, \mathrm{d}x.$$

for all $\sigma_1, \sigma_2 \in L_+^\infty(\Omega)$, $g \in L_\diamond^2(\partial\Omega)$.

- Monotonicity w.r.t. Loewner order:

$$\sigma_1 \leq \sigma_2 \quad \Longrightarrow \quad \Lambda(\sigma_1) \geq \Lambda(\sigma_2)$$

  (Loewner order for self-adjoint operators: $A \geq B$ iff $A - B$ is positive semi-definite)

- Localized potentials: (**H.**, 2008)

$$\exists (g_k)_{k\in\mathbb{N}} : \int_{D_1} |\nabla u_\sigma^{g_k}|^2 \, \mathrm{d}x \to \infty, \quad \int_{D_2} |\nabla u_\sigma^{g_k}|^2 \, \mathrm{d}x \to 0.$$

  (if D_1 can be reached without crossing D_2)

$\rightsquigarrow$ Monotonicity method for inclusion detection

Monotonicity method for inclusion detection

- EIT: Detect $\sigma \in L_+^\infty(\Omega)$ in $\nabla \cdot (\sigma \nabla u) = 0$ from NtD $\Lambda(\sigma)$
- Inclusion detection (simplest case):

$$\sigma = 1 + \chi_D, \quad D \text{ open, with connected complement}$$

- Monotonicity:

$$\sigma_1 \leq \sigma_2 \quad \Longrightarrow \quad \Lambda(\sigma_1) \geq \Lambda(\sigma_2)$$

- Monotonicity & Localized Potentials:
  *(„$\Longrightarrow$": Tamburrino/Rubinacci 2002, „$\Longleftarrow$" & Linearization: **H.**/Ullrich 2013)*

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1+\chi_B) \geq \Lambda(1+\chi_D)$$

$$\Longleftrightarrow \quad \Lambda(1) + \tfrac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(1+\chi_D)$$

Inclusion can be found by testing several small balls B

Monotonicity and Convexity

Lemma.

$$\int_{\partial\Omega} g(\Lambda(\sigma_1)-\Lambda(\sigma_2))g \,\mathrm{d}s \geq \int_\Omega (\sigma_2-\sigma_1)|\nabla u^g_{\sigma_2}|^2 \,\mathrm{d}x$$
$$= \int_{\partial\Omega} g\Lambda'(\sigma_2)(\sigma_1-\sigma_2)g \,\mathrm{d}s.$$

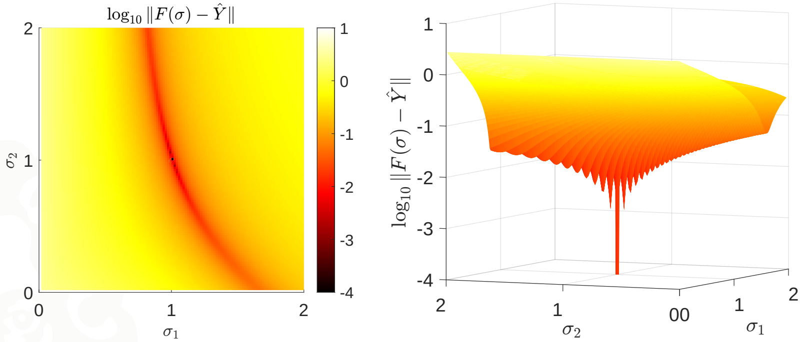
for all $\sigma_1, \sigma_2 \in L^\infty_+(\Omega)$, $g \in L^2_\diamond(\partial\Omega)$.

- ⇝ For all $\sigma_1, \sigma_2 \in L^\infty_+(\Omega)$: $\Lambda(\sigma_1)-\Lambda(\sigma_2) \geq \Lambda'(\sigma_2)(\sigma_1-\sigma_2)$.
- ⇝ Convexity: For all $\sigma_1, \sigma_2 \in L^\infty_+(\Omega)$, $t \in [0,1]$

$$\Lambda((1-t)\sigma_1 + t\sigma_2) \leq (1-t)\Lambda(\sigma_1) + t\Lambda(\sigma_2).$$

The "monotonicity lemma" also implies convexity.

Convex or not?



- Standard data-fitting yields non-convex residuum functionals
- But the NtD is convex w.r.t. Loewner order...

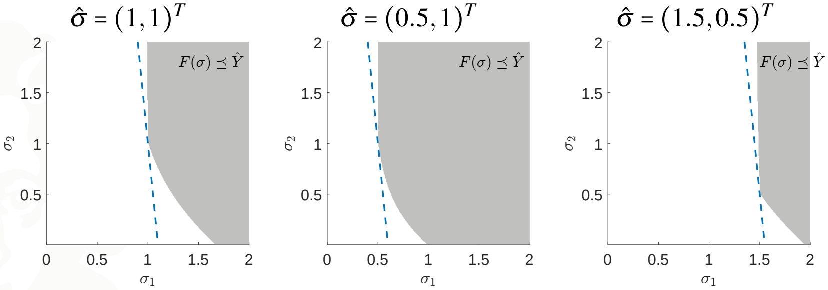
Can we find convex reformulation of the Calderón problem?

Convexity for the simple example

Inverse problem: Reconstruct $\hat{\sigma} \in \mathbb{R}^2_+$ from $\hat{Y} = F(\hat{\sigma}) \in \mathbb{R}^{6\times 6}$

- F monoton. non-increasing and convex w.r.t. Loewner order

$\rightsquigarrow$ $\{\sigma : F(\sigma) \preceq \hat{Y}\} \subset \mathbb{R}^2$ convex set

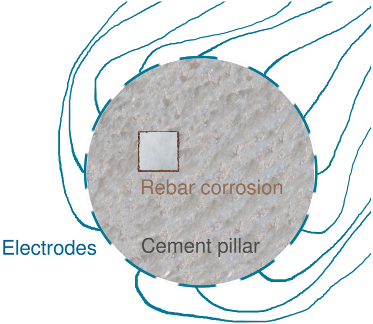


Observation.

$\hat{\sigma}$ is the lower left corner of the convex set $F(\sigma) \preceq \hat{Y}$.

An inverse Robin coefficient problem

EIT for corrosion detection



Non-destructive EIT-based corrosion detection:

- Apply electric currents on outer boundary $\partial\Omega$
- Measure necessary voltages

$\leadsto$ Detect corrosion on inner boundary $\Gamma = \partial D$

Idealized mathematical model: Robin PDE

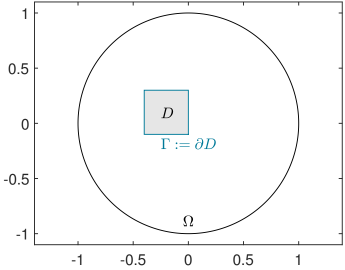
Electric potential $u: \Omega \to \mathbb{R}$ solves

(1) $\Delta u = 0$ in $\Omega \setminus \Gamma$,

(2) $\partial_\nu u|_{\partial\Omega} = g$ on $\partial\Omega$,

(3) $[\![u]\!]_\Gamma = 0$ on Γ,

(4) $[\![\partial_\nu u]\!]_\Gamma = \sigma u$ on Γ



Inverse Problem: Recover σ from Neumann-to-Dirichlet-Operator

$$\Lambda(\sigma): L^2(\partial\Omega) \to L^2(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves Robin PDE (1)–(4).

Finitely many measurements and unknowns

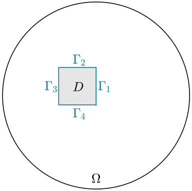
- Finitely many measurements:

$$\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, \mathrm{d}s \quad \text{for finitely many } g_1, \ldots, g_m$$

- Finite desired resolution:

$$\sigma = \sum_{j=1}^{n} \sigma_j \chi_{\Gamma_j} \quad \text{with } \sigma_j \in \mathbb{R},\ j = 1, \ldots, n$$

with partition $\Gamma = \bigcup_{j=1}^{n} \Gamma_j$



- A-priori bounds: $\sigma := (\sigma_1, \ldots, \sigma_n)^T \in [a,b]^n$, $b > a > 0$ known

Finite-dimensional non-linear inverse problem: Determine

$$\sigma = (\sigma_j)_{j=1}^{n} \in [a,b]^n \quad \text{from} \quad F(\sigma) := \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, \mathrm{d}s \right)_{j,k=1}^{m} \in \mathbb{R}^{m \times m}$$

Convex formulation of inverse Robin problem 1/3

Theorem. (**H.**, Optim. Lett. 2021)
If sufficiently many measurements are taken, then

- $\hat{Y} := F(\hat{\sigma}) \in \mathbb{R}^{m\times m}$ uniquely determines $\hat{\sigma} \in [a,b]^n$.
- $\hat{\sigma}$ is the unique solution of

$$\texttt{minimize} \quad \|\sigma\|_1 = \sum_{j=1}^{n} \sigma_j \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \leq \hat{Y}.$$

- The constraint set $\sigma \in [a,b]^n,\ F(\sigma) \leq \hat{Y}$ is convex.

⇝ $\hat{\sigma}$ is the lower left corner of the convex constraint set $F(\sigma) \leq \hat{Y}$

⇝ Problem can be solved by convex semidefinite programming

Global convergence is feasible.

(**H.**, Numer. Math. 2020: Global Newton convergence for this Robin problem)

Convex formulation of inverse Robin problem 2/3

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Theorem. (**H.**, Optim. Lett. 2021)

- Suff. many measurements are taken if $\lambda_{\text{max}}(F'(z_{j,k})d_j) > 0$ for
  $$z_{j,k} := \frac{a}{2}e'_j + \left(a + k\frac{a}{4(n-1)}\right)e_j \in \mathbb{R}^n_+, \quad d_j := \frac{2b-a}{a}(n-1)e'_j - \frac{1}{2}e_j \in \mathbb{R}^n,$$
  with $j = 1, \ldots, n, \quad k = 1, \ldots, \lceil \frac{4(n-1)b}{a} \rceil - 4n + 5.$
- This criterion is fulfilled if $(g_j)_{j=1}^{\infty}$ has dense span in $L^2(\partial\Omega)$, and sufficiently many g_j are used.

($e_j \in \mathbb{R}^n$: j-th unit vector, $e'_j := \mathbb{1} - e_j \in \mathbb{R}^n$: negated j-th unit vector)

$\leadsto$ Explicit, easy-to-check criterion whether a desired resolution can be achieved with a certain number of measurements

Achievable resolution can be characterized.

Convex formulation of inverse Robin problem 3/3

Theorem. (**H.**, Optim. Lett. 2021)

- Let the criterion hold with lower bound $\lambda > 0$.
- Let $\delta > 0$, and $Y^\delta \in \mathbb{R}^{m\times m}$ be symmetric with $\|\hat{Y} - Y^\delta\|_2 \leq \delta$.

Then there exist solutions of

$$\texttt{minimize} \quad \|\sigma\|_1 = \sum_{j=1}^{n} \sigma_j \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \leq Y^\delta + \delta I.$$

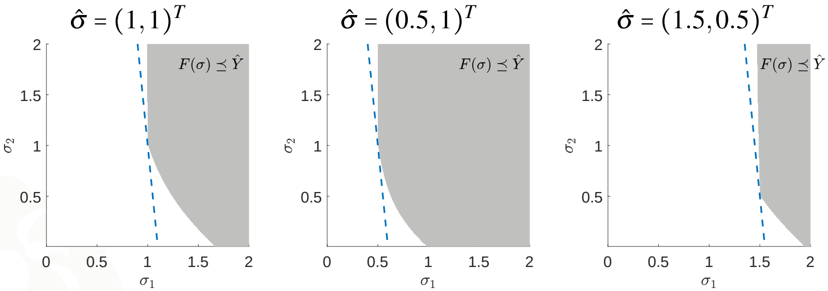
and every such minimum σ^δ fulfills

$$\|\hat{\sigma} - \sigma^\delta\|_\infty \leq \frac{2\delta(n-1)}{\lambda}$$

Explicit error estimates, convergence for $\delta \to 0$.

Back to Electrical Impedance Tomography

The lower left corner for EIT



- $\hat{\sigma}$ not minimizer of unweighted sum $\|\sigma\|_1 = \sum_{i=1}^n \sigma_i \to$ min!

Conjecture. $\exists c \in \mathbb{R}^n$ so that true solution $\hat{\sigma}$ minimizes

$$c^T \sigma = \sum_{i=1}^{n} c_i \sigma_i \to \text{min!} \quad \text{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \leq \hat{Y}.$$

Convex reformulation for EIT

Theorem. *(**H.**, SIMA 2023)*
If sufficiently many measurements are taken, then:

- EIT forward mapping $F: [a,b]^n \to \mathbb{S}_m \subset \mathbb{R}^{m\times m}$ is injective.
- Derivative $F'(\sigma)$ is injective for all $\sigma \in [a,b]^n$.
- There exists $c \in \mathbb{R}^n_+$ so that for all $\hat{\sigma} \in [a,b]^n$, $\hat{Y} = \Lambda(\hat{\sigma})$:

  $\hat{\sigma}$ is the unique solution of the convex problem

$$\texttt{minimize} \quad c^T\sigma = \sum_{i=1}^n c_i\sigma_i \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \preceq \hat{Y}.$$

The Calderón problem with finitely many unknowns is equivalent to convex semidefinite optimization

Stability and error estimates

Theorem (continued). *(**H.**, SIMA 2023)*
There exists $\lambda > 0$ so that

- for all $\hat{\sigma} \in [a,b]^n$, and $\hat{Y} := \Lambda(\hat{\sigma})$,
- and all $\delta > 0$, and $Y^\delta \in \mathbb{S}_m \subset \mathbb{R}^{m\times m}$, with $\|Y^\delta - \hat{Y}\| \leq \delta$,

the convex semidefinite optimization problem

$$\texttt{minimize} \quad c^T\sigma = \sum_{i=1}^n c_i\sigma_i \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \leq Y^\delta + \delta I.$$

possesses a minimizer σ^δ. Every such minimizer fulfills

$$\|\sigma^\delta - \hat{\sigma}\|_{c,\infty} \leq \frac{n-1}{\lambda}\delta.$$

($\|\cdot\|_{c,\infty}$: c-weighted maximum norm)

Error estimates for noisy data $Y^\delta \approx \hat{Y}$ also hold.

Conclusions 1/2

For elliptic coefficient inverse problems

- least-squares residuum functionals may be highly non-convex
- local minima are usually useless

Possible remedy

- utilize monotonicity & convexity with respect to Loewner order
- utilize localized potentials to control directional derivatives

Equivalent convex reformulations are possible

- globally convergent solution algorithms are possible
- error estimates for noisy data are possible
- For similar but simpler Robin problem
  - explicit characterizations of achievable resolution
  - explicit error estimates for noisy data

Conclusions 2/2 *(now getting very subjective)*

Some future challenges in inverse problems in PDEs:

We need to progress and extend. . .

FROM: Uniqueness results for infinite-dimensional DtN/NtD
TO: Resolution attainable from finitely many measurements

FROM: Stability results
TO: Error estimates (with computable constants)

FROM: Local convergence and non-convex residuum functionals
TO: Global convergence and convex functionals

Loewner monotonicity & convexity can help with these challenges.