

Uniqueness and convex reformulation for inverse coefficient problems with finitely many measurements

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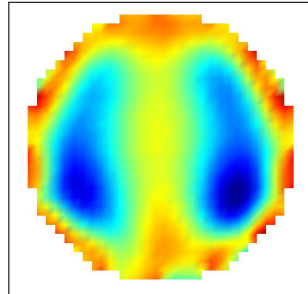
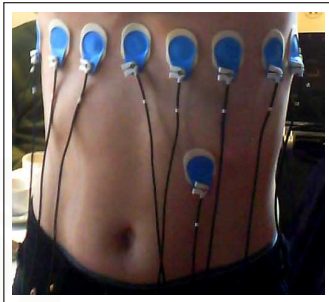
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EIT with finitely many measurements

Electrical impedance tomography (EIT)



- ▶ Apply electric currents on subject's boundary
- ▶ Measure necessary voltages
- ➞ Reconstruct conductivity inside subject

Calderón problem

Can we recover $\sigma \in L_+^\infty(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \subset \mathbb{R}^d \quad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves (1)}\} ?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma) : L_\diamond^2(\partial\Omega) \rightarrow L_\diamond^2(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Challenges in idealized EIT

Mathematical idealization of EIT \leadsto Calderón problem

- ▶ infinitely many unknowns $\sigma \in L_+^\infty(\Omega)$
- ▶ infinitely many measurements $\Lambda(\sigma) \in \mathcal{L}(L_\diamond^2(\partial\Omega))$
- ▶ nonlinear forward map $\sigma \mapsto \Lambda(\sigma)$

Mathematical challenges

- ▶ Uniqueness? Does $\Lambda(\sigma)$ determine σ ?
- ▶ Stability? $\Lambda^{-1} : \Lambda(\sigma) \mapsto \sigma$ continuous?
- ▶ Convergence (local/global)? How to determine σ from $\Lambda(\sigma)$?

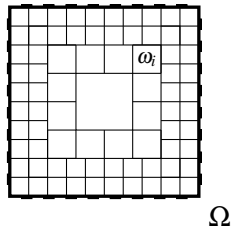
Consequences for practical EIT?

EIT in practice

- ▶ Finitely many unknowns, σ pcw. const. on given resolution $\Omega = \bigcup_{i=1}^n \omega_i$
- ▶ Finitely many measurements

$$\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, ds$$

for given currents $g_1, \dots, g_m \in L^2_{\diamond}(\partial\Omega)$



Finite-dimensional inverse problem: Determine

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix} \in \mathbb{R}_+^n \quad \text{from } F(\sigma) = \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, ds \right)_{j,k=1}^m \in \mathbb{R}^{m \times m}.$$

Mathematical challenges for practical EIT

Inverse problem: Determine $\sigma \in \mathbb{R}_+^n$ from $Y = F(\sigma) \in \mathbb{R}^{m \times m}$.

For a fixed desired resolution:

- ▶ How many measurements uniquely determine σ ?
- ▶ Stability / error estimates for noisy data $Y^\delta \approx F(\sigma)$?
- ▶ Numerical algorithm to determine $\sigma \in \mathbb{R}_+^n$ from $Y^\delta \approx F(\sigma)$?
- ▶ Global/local convergence of algorithm?

This talk: The problem of local convergence, a bold guess, and answers for a Robin problem (similar to but simpler than EIT)

The problem of local minima and a bold guess

Simple example: EIT with 2 unknowns & 6 bndry. currents

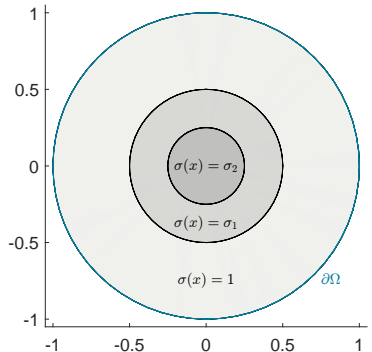
Ω : unit circle

$$F: \mathbb{R}_+^2 \rightarrow \mathbb{R}^{6 \times 6}$$

$$F \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} := \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \right)_{j,k=1}^6$$

with trigonometric currents

$$\{g_1, \dots, g_6\} = \{\sin(\varphi), \dots, \cos(3\varphi)\}$$

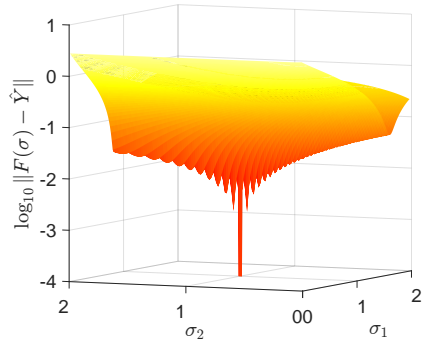
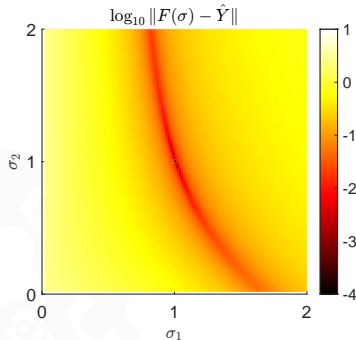


Inverse problem: Reconstruct $\hat{\sigma} \in \mathbb{R}_+^2$ from $\hat{Y} = F(\hat{\sigma}) \in \mathbb{R}^{6 \times 6}$

Natural approach: Least squares data fitting

$$\text{minimize} \quad \|F(\sigma) - \hat{Y}\|_F^2 \quad (+ \text{Regularization})$$

Problem of local minima



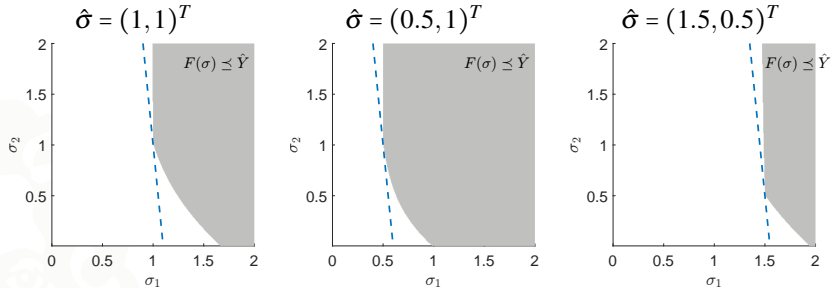
Numerical results indicate

- ▶ $\hat{Y} = F(\hat{\sigma})$ uniquely determines $\hat{\sigma} \dots$
- ▶ \dots but residuum is highly non-convex, many local minima

Are globally convergent algorithms impossible?

Bold guess

Inverse problem: Reconstruct $\hat{\sigma} \in \mathbb{R}_+^2$ from $\hat{Y} = F(\hat{\sigma}) \in \mathbb{R}^{6 \times 6}$



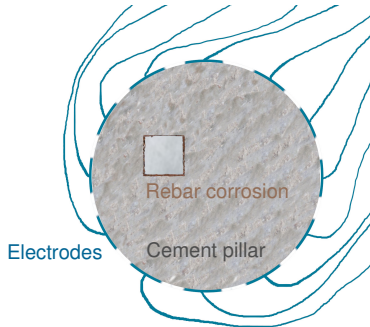
Bold conjecture.

$\hat{\sigma}$ is the lower left corner of the convex set $F(\sigma) \preceq \hat{Y}$.

("≤": Loewner / semidefiniteness order)

An inverse Robin coefficient problem

EIT for corrosion detection



Non-destructive EIT-based corrosion detection:

- ▶ Apply electric currents on outer boundary $\partial\Omega$
- ▶ Measure necessary voltages
- Detect corrosion on inner boundary $\Gamma = \partial D$

Idealized mathematical model: Robin PDE

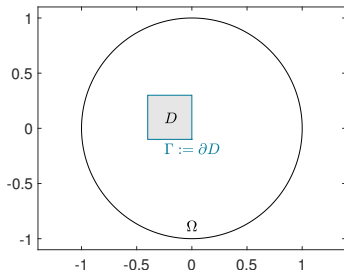
Electric potential $u: \Omega \rightarrow \mathbb{R}$ solves

$$(1) \quad \Delta u = 0 \quad \text{in } \Omega \setminus \Gamma,$$

$$(2) \quad \partial_\nu u|_{\partial\Omega} = g \quad \text{on } \partial\Omega,$$

$$(3) \quad \llbracket u \rrbracket_\Gamma = 0 \quad \text{on } \Gamma,$$

$$(4) \quad \llbracket \partial_\nu u \rrbracket_\Gamma = \sigma u \quad \text{on } \Gamma$$



Inverse Problem: Recover σ from Neumann-to-Dirichlet-Operator

$$\Lambda(\sigma): L^2(\partial\Omega) \rightarrow L^2(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves Robin PDE (1)–(4).

Finitely many measurements and unknowns

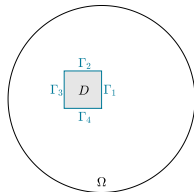
- ▶ Finitely many measurements:

$$\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, ds \quad \text{for finitely many } g_1, \dots, g_m$$

- ▶ Finite desired resolution:

$$\sigma = \sum_{j=1}^n \sigma_j \chi_{\Gamma_j} \quad \text{with } \sigma_j \in \mathbb{R}, j = 1, \dots, n$$

with partition $\Gamma = \bigcup_{j=1}^n \Gamma_j$



- ▶ A-priori bounds: $\sigma := (\sigma_1, \dots, \sigma_n)^T \in [a, b]^n$, $b > a > 0$ known

Finite-dimensional non-linear inverse problem: Determine

$$\sigma = (\sigma_j)_{j=1}^n \in [a, b]^n \quad \text{from} \quad F(\sigma) := \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, ds \right)_{j,k=1}^m \in \mathbb{R}^{m \times m}$$

Main result 1/3

Theorem. (H., Optim. Lett. 2021)

If sufficiently many measurements are taken, then

- ▶ $\hat{Y} := F(\hat{\sigma}) \in \mathbb{R}^{m \times m}$ uniquely determines $\hat{\sigma} \in [a, b]^n$.
- ▶ $\hat{\sigma}$ is the unique solution of

$$\text{minimize } \|\sigma\|_1 = \sum_{j=1}^n \sigma_j \quad \text{s.t.} \quad \sigma \in [a, b]^n, F(\sigma) \leq \hat{Y}.$$

- ▶ The constraint set $\sigma \in [a, b]^n, F(\sigma) \leq \hat{Y}$ is convex.

- $\hat{\sigma}$ is the lower left corner of the convex constraint set
- Problem can be solved by convex semidefinite programming

Global convergence is feasible.

(H., Numer. Math. 2020: Global Newton convergence for this Robin problem)

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Main result 2/3

Theorem. (H., Optim. Lett. 2021)

- ▶ Suff. many measurements are taken if $\lambda_{\max}(F'(z_{j,k})d_j) > 0$ for $z_{j,k} := \frac{a}{2}e'_j + \left(a + k\frac{a}{4(n-1)}\right)e_j \in \mathbb{R}^n$, $d_j := \frac{2b-a}{a}(n-1)e'_j - \frac{1}{2}e_j \in \mathbb{R}^n$, with $j = 1, \dots, n$, $k = 1, \dots, \lceil \frac{4(n-1)b}{a} \rceil - 4n + 5$.
- ▶ This **criterion** is fulfilled if $(g_j)_{j=1}^{\infty}$ has dense span in $L^2(\partial\Omega)$, and sufficiently many g_j are used.

($e_j \in \mathbb{R}^n$: j -th unit vector, $e'_j := \mathbb{1} - e_j \in \mathbb{R}^n$: negated j -th unit vector)

- ↪ Explicit, easy-to-check criterion whether a desired resolution can be achieved with a certain number of measurements

Achievable resolution can be characterized.

Main result 3/3

Theorem. (H., Optim. Lett. 2021)

- ▶ Let the **criterion** hold with lower bound $\lambda > 0$.
- ▶ Let $\delta > 0$, and $Y^\delta \in \mathbb{R}^{m \times m}$ be symmetric with $\|\hat{Y} - Y^\delta\|_2 \leq \delta$.

Then there exist solutions of

$$\text{minimize } \|\sigma\|_1 = \sum_{j=1}^n \sigma_j \quad \text{s.t.} \quad \sigma \in [a, b]^n, F(\sigma) \leq Y^\delta + \delta I.$$

and every such minimum σ^δ fulfills

$$\|\hat{\sigma} - \sigma^\delta\|_\infty \leq \frac{2\delta(n-1)}{\lambda}$$

Explicit error estimates, convergence for $\delta \rightarrow 0$.

Proof ingredients & possible generalizations

- **Monotonicity & Convexity:** $F : \mathbb{R}_+^n \rightarrow \mathbb{S}_m \subset \mathbb{R}^{m \times m}$ fulfills

$$\begin{aligned} F'(\sigma)d &\leq 0 && \text{for all } \sigma \in \mathbb{R}_+^n, 0 \leq d \in \mathbb{R}^n \\ F(\tau) - F(\sigma) &\geq F'(\sigma)(\tau - \sigma) && \text{for all } \sigma, \tau \in \mathbb{R}_+^n \end{aligned}$$

↪ holds for general elliptic PDEs (H., Jahresber. DMV, 2021)

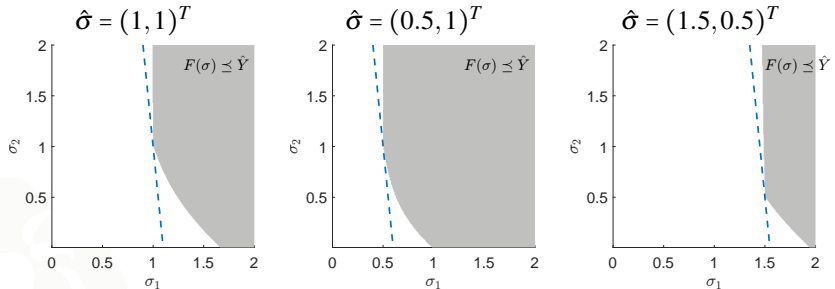
- **Localized potentials:** For any $C > 0$, there exist currents g s.t.

$$g^T (F'(\sigma)(e_j - Ce'_j))g = \int_{\Gamma_j} |\nabla u|^2 \, dx - C \int_{\Gamma \setminus \Gamma_j} |\nabla u|^2 > 0$$

$$\implies \lambda_{\max}(F'(z)(e_j - Ce'_j)) > 0 \text{ for suff. many measurement.}$$

↪ holds for many elliptic problems, but in more complicated form

Back to the bold guess in EIT



- **Robin problem:** $\hat{\sigma}$ minimizes

$$\mathbb{1}^T \sigma = \sum_{i=1}^n \sigma_i \rightarrow \min! \quad \text{s.t.} \quad \sigma \in [a, b]^n, F(\sigma) \leq \hat{Y}.$$

- **EIT:** Can we find $c \in \mathbb{R}^n$ so that $\hat{\sigma}$ minimizes

$$c^T \sigma = \sum_{i=1}^n c_i \sigma_i \rightarrow \min! \quad \text{s.t.} \quad \sigma \in [a, b]^n, F(\sigma) \leq \hat{Y}?$$

Convex reformulation for EIT

Theorem. (H., arxiv:2203.16779, submitted last week)

If sufficiently many measurements are taken, then:

- ▶ EIT forward mapping $F : [a, b]^n \rightarrow \mathbb{S}_m \subset \mathbb{R}^{m \times m}$ is injective.
- ▶ Derivative $F'(\sigma)$ is injective for all $\sigma \in [a, b]^n$.
- ▶ There exists $c \in \mathbb{R}_+^n$ so that for all $\hat{\sigma} \in [a, b]^n$, $\hat{Y} = \Lambda(\hat{\sigma})$:

$\hat{\sigma}$ is the unique solution of the convex problem

$$\text{minimize } c^T \sigma = \sum_{i=1}^n c_i \sigma_i \quad \text{s.t.} \quad \sigma \in [a, b]^n, F(\sigma) \preceq \hat{Y}.$$

(Error estimates for noisy data $Y^\delta \approx \hat{Y}$ also hold.)

The Calderón problem with finitely many unknowns is equivalent to convex semidefinite optimization

Conclusions

For elliptic coefficient inverse problems

- ▶ least-squares residuum functionals may be highly non-convex
- ▶ local minima are usually useless

Possible remedy

- ▶ utilize monotonicity & convexity with respect to Loewner order
- ▶ utilize localized potentials to control directional derivatives

Equivalent convex reformulations are possible

- ▶ globally convergent solution algorithms are possible
- ▶ error estimates for noisy data are possible
- ▶ For simple Robin problem
 - ▶ explicit characterizations of achievable resolution
 - ▶ explicit error estimates for noisy data