Monotonicity-Based Inversion of the Fractional Schrödinger Equation

Bastian Harrach
(joint work with Yi-Hsuan Lin)

http://numerical.solutions

Institute of Mathematics, Goethe University Frankfurt, Germany

2022 SIAM Conference on Imaging Science (IS22)
Berlin, Germany, March 21–25, 2022.

Fractional Schrödinger/Helmholtz equation

- Dirichlet problem in Lipschitz bounded open set $\Omega \subset \mathbb{R}^n$

  $(n \in \mathbb{N},\ 0 < s < 1,\ q \in L^\infty(\Omega))$

  (1) $$(-\Delta)^s u + qu = 0 \quad \text{in } \Omega$$

  (2) $$u|_{\Omega_e} = F \quad \text{in } \Omega_e := \mathbb{R}^n \setminus \Omega$$

- Dirichlet-to-Neumann-operator

  $$\Lambda(q):\ H(\Omega_e) \to H(\Omega_e)^*, \quad F \mapsto (-\Delta)^s u|_{\Omega_e},$$

  where u solves (1), (2).

  ($H(\Omega_e) := H^s(\mathbb{R}^n)/\overline{C_c^\infty(\Omega)}$ or subspace of finite codimension to avoid resonances)

Fractional Calderón problem: Can we recover q from $\Lambda(q)$?

Literature

Known results on uniqueness and stability:

- Uniqueness for fract. Calderón problem: *Ghosh/Salo/Uhlmann '16*
- Uniqueness with one measurement: *Ghosh/Rüland/Salo/Uhlmann '18*
- Logarithmic stability results: *Rüland/Salo '17+'18*
- Lipschitz stability in finite dim. from specific finitely many measurements (depending on unknown q): *Rüland/Sincich '18*

This talk: Monotonicity method for fractional Calderón problem

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$\rightsquigarrow$ Constructive uniqueness proof for fractional Calderón problem

$\rightsquigarrow$ Reconstruction method for inclusion detection problems

$\rightsquigarrow$ Lipschitz stability in finite dimensions from arbitrary (but sufficiently many) measurements

Monotonicity method for positive potentials $q \in L^\infty_+(\Omega)$

Monotonicity for positive potentials

Theorem. For two potentials $q_0, q_1 \in L^\infty_+(\Omega)$:

$$q_0 \leq q_1 \quad \Longleftrightarrow \quad \Lambda(q_0) \preceq \Lambda(q_1)$$

$q_0 \leq q_1$ *understood pointwise (a.e.),* $\Lambda(q_0) \preceq \Lambda(q_1)$ *w.r.t. Loewner order)*

Sketch of proof:

- Use variational formulation (as in *Kang/Seo/Sheen 1997, Ikehata 1998*):

$$\int_\Omega (q_1 - q_0)|u_0|^2 \geq \int_{\partial\Omega} F\,(\Lambda(q_1) - \Lambda(q_0))\,F \geq \int_\Omega \frac{q_0}{q_1}(q_1 - q_0)|u_0|^2$$

- Derive localized potentials (from UCP in *Ghosh/Salo/Uhlmann 2016*):

$$\forall \text{meas. } M \subseteq \Omega: \ \exists (F^k)_{k\in\mathbb{N}}: \ \int_M |u^k|^2 \to \infty, \quad \int_{\Omega\setminus M} |u^k|^2 \to 0.$$

Monotonicity-based uniqueness

Corollary. $q \in L^\infty_+(\Omega)$ can be recovered from $\Lambda(q)$ via

$$q(x) = \sup\{\psi(x): \ \psi \in \Sigma_+, \ \Lambda(\psi) \leq \Lambda(q)\} \quad \forall x \in \Omega \text{ (a.e.)}$$

Σ_+: space of *density one* simple functions with positive infima

(*Density one*: only zero value is attained on a null set)

Sketch of proof: Show that for all $q \in L^\infty_+(\Omega)$

$$q(x) = \sup\{\psi(x): \ \psi \in \Sigma_+, \ \psi \leq q\} \quad \forall x \in \Omega \text{ (a.e.)}$$

and apply monotonicity.

Implementation possible but requires many forward solutions $\Lambda(\psi)$

Fast monotonicity-based inclusion detection

Theorem. For two potentials $q_0, q_1 \in L^\infty_+(\Omega)$:

$$\operatorname{supp}(q_1 - q_0) = \left\{ \begin{array}{l} \text{intersection of all closed sets } C \subseteq \Omega \text{ with} \\ \exists \alpha > 0: \ -\alpha \Lambda'(q_0)\chi_C \leq \Lambda(q_1) - \Lambda(q_0) \leq \alpha \Lambda'(q_0)\chi_C \end{array} \right.$$

Sketch of proof: Use that

$$\int_{\partial\Omega} F \left(\Lambda(q_0)' \chi_C \right) F \, \mathrm{d}s = \int_C |u_0|^2 \, \mathrm{d}x$$

and apply monotonicity and localized potentials result.

Reconstructing where unknown potential q_1 differs from known q_0 requires only one forward solution for q_0.

Monotonicity and stability for general potentials $q \in L^\infty(\Omega)$

Monotonicity for general potentials

Define modified Loewner order:

$$\Lambda(q_0) \preceq_{\text{fin}} \Lambda(q_1) \quad :\Longleftrightarrow \quad \int_{\partial\Omega} F\,(\Lambda(q_1) - \Lambda(q_0))\,F \geq 0$$

on subspace with finite codimension in $H(\Omega_e)$.

(In case of resonances, $\Lambda(q_1)$, $\Lambda(q_0)$ may be defined on different fin. codim. subspaces)

Theorem. For two potentials $q_0, q_1 \in L^\infty(\Omega)$:

$$q_0 \leq q_1 \quad \Longleftrightarrow \quad \Lambda(q_0) \preceq_{\text{fin}} \Lambda(q_1)$$

. . . and finite codimension is bounded by function $d(q_0)$

Sketch of proof:

Use compact perturbation ideas (similar to *H./Pohjola/Salo, Anal. PDE 2019*).

Monotonicity-based uniqueness

Corollary. $q \in L^\infty(\Omega)$ can be recovered from $\Lambda(q)$ via

$$q(x) = \sup\{\psi(x):\ \psi \in \Sigma,\ \Lambda(\psi) \preceq_{\text{fin}} \Lambda(q)\} + \inf\{\psi(x):\ \psi \in \Sigma,\ \Lambda(\psi) \succeq_{\text{fin}} \Lambda(q)\} \quad \forall x \in \Omega \text{ (a.e.)}$$

Σ: space of *density one* simple functions

(*Density one*: only zero value is attained on a null set)

Sketch of proof: Show that for all $q \in L^\infty(\Omega)$

$$\max\{q(x),0\} = \sup\{\psi(x):\ \psi \in \Sigma,\ \psi \leq q\} \quad \forall x \in \Omega \text{ (a.e.)}$$

and apply monotonicity, and

$$q(x) = \max\{q(x),0\} - \max\{-q(x),0\}.$$

(Note that $\max$ *is required since density one simple functions can be zero on null sets.)*

Fast monotonicity-based inclusion detection

Theorem. For two non-resonant potentials $q_0, q_1 \in L^\infty_+(\Omega)$:

$$\operatorname{supp}(q_1 - q_0) = \left\{ \begin{array}{l} \text{intersection of all closed sets } C \subseteq \Omega \text{ where } \exists \alpha > 0: \\ -\alpha \Lambda'(q_0)\chi_C \preceq_{\text{fin}} \Lambda(q_1) - \Lambda(q_0) \preceq_{\text{fin}} \alpha \Lambda'(q_0)\chi_C \end{array} \right.$$

Sketch of proof:

- Harder than before, since no known analogue for

$$\int_{\partial\Omega} F\left(\Lambda(q_1) - \Lambda(q_0)\right) F \geq \int_\Omega \frac{q_0}{q_1}(q_1 - q_0)|u_0|^2.$$

- Requires *simultaneously* localized potent. on fin. codim. spaces: $\forall q_0, q_1 \in L^\infty(\Omega)$, $\operatorname{supp}(q_1 - q_0) \subseteq M \subseteq \Omega$ meas., $\exists (F_k)_{k \in \mathbb{N}}$:

$$\int_M |u_0^k|^2 \to \infty, \quad \int_{\Omega \setminus M} |u_0^k|^2 \to 0$$
$$\int_M |u_1^k|^2 \to \infty, \quad \int_{\Omega \setminus M} |u_1^k|^2 \to 0$$

Uniqueness & stability with finitely many measurements

- $\mathcal{Q} \subset L^\infty(\Omega)$ fin.-dim. subspace, $\mathcal{Q}_a := \{q \in \mathcal{Q} : \ \|q\|_\infty \leq a\}$
- Sequence of (e.g., fin.-dim.) subspace

$$H_1 \subseteq H_2 \subseteq \ldots \subseteq H(\Omega_e), \quad \overline{\bigcup_{l \in \mathbb{N}} H_l} = H(\Omega_e)$$

Theorem. There exists $k \in \mathbb{N}$ and $c > 0$:

$$\|P'_{H_l}(\Lambda(q_2) - \Lambda(q_1))P_{H_l}\| \geq \frac{1}{c}\|q_2 - q_1\| \quad \forall q_1, q_2 \in \mathcal{Q}_a,\ l \geq k.$$

P_{H_l}: Galerkin projection to H_l (or fin. codim. space in resonant case)

Sketch of proof: Use ideas from monotonicity-based stability proofs (from ***H.**/Meftahi, SIAP 2019* and ***H.**, IP 2019*)

Summary

$$q_0 \leq q_1 \quad \Longleftrightarrow \quad \Lambda(q_0) \preceq_{\text{fin}} \Lambda(q_1).$$

Monotonicity-based ideas yield

- constructive uniqueness result,
- fast inclusion detection methods,
- uniqueness stability for finitely many measurements,

Outlook:

- Equivalent formulation as convex semidef. optimization probl.?

References:

- **H.**/Lin: Monotonicity-based inversion of the fractional Schrödinger equation I. Positive potentials, *SIMA* 2019
- **H.**/Lin: Monotonicity-based inversion of the fractional Schrödinger equation II. General potentials and stability, *SIMA* 2019
- **H.** Solving an inverse elliptic coefficient problem by convex non-linear semidefinite programming *Optim. Lett.* 2021.