Stable Differentiation of Monte Carlo Priced Options with Discontinuous Payoff

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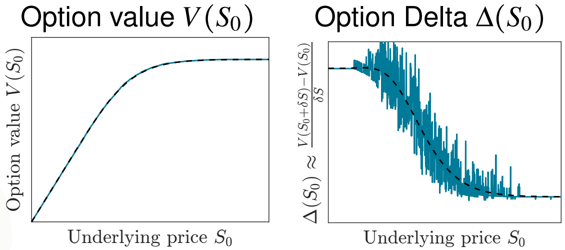
Motivation

Monte Carlo pricing of option with discontinuous payoff

(here: autocallable, aka express-certificate or barrier-option)



Standard Monte Carlo (blue line)

- works well for pricing
- fails for calculating derivatives, e.g., $\Delta(S_0) = \frac{\partial}{\partial S} V(S_0)$

Can we stably calculate greeks of discontinuous payoff options?

The problem of unstable greeks

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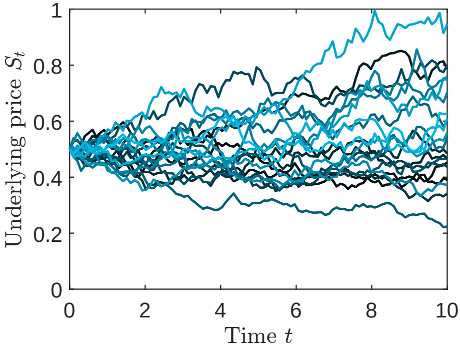
Monte Carlo Option Pricing



Today's option value $V(S_0)$

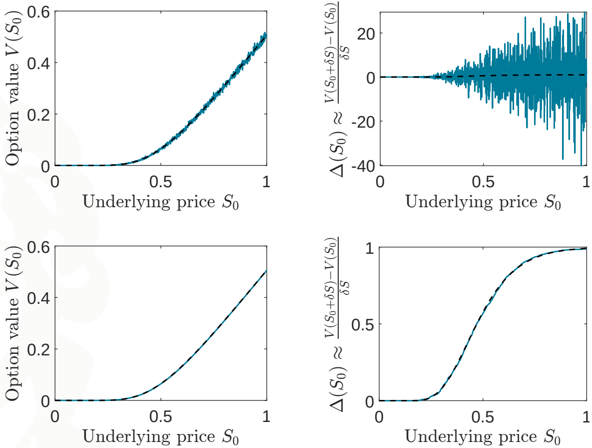
$=$ (discounted) expected payoff at maturity

$\approx$ (discounted) mean payoff of randomly sampled paths

Simple example for instability

European “plain vanilla” call option: $V(S_T) = \max\{S_T - K, 0\}$

Option value $V(S_0)$ | Option Delta $\Delta(S_0)$

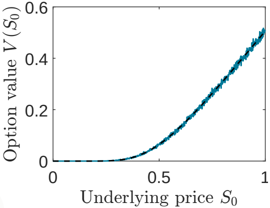


w/o resetting random seed

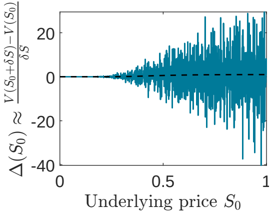
with resetting random seed

MC pricing without resetting random seed

Option value $V(S_0)$



Option Delta $\Delta(S_0)$



When #MC samples $\to \infty$:

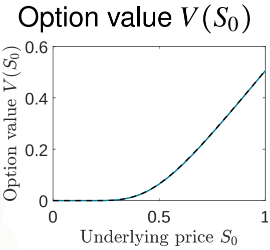
- MC approximations of $V(S_0)$ converge against true $V(S_0)$
- MC approximations of $\Delta(S_0)$ do not converge

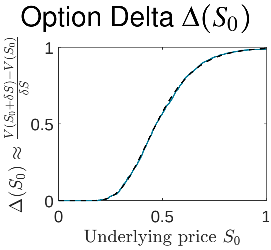
Mathematical background:

Convergence of function values $\not\Longrightarrow$ Convergence of derivative(s)

Differentiation is intrinsically unstable.

MC pricing with resetting random seed





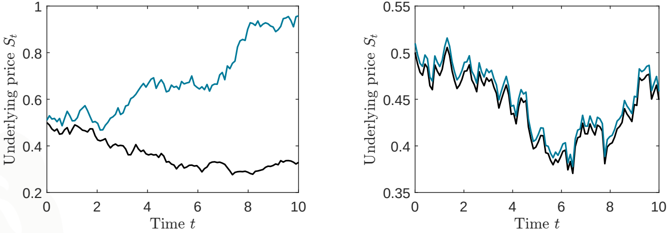
When #MC samples $\to \infty$:

- Function values, and derivative(s) converge.

MC pricing with convergence of values and derivative(s) is possible.

Reason for stable derivatives

Random walk starting with S_0 and with $S_0 + \delta S$:



w/o resetting random seed   with resetting random seed

With resetting random seed:

- Final path value S_T depends Lipschitz continuously on S_0
- European Option Payoff depends Lipschitz continuously on S_T

$\rightsquigarrow$ Path payoff depends Lipschitz continuously on S_0

Stable greeks for continuous payoff options

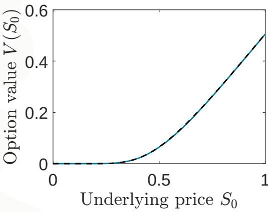
Alm/**H.**/Harrach/Keller, JCF 2013:

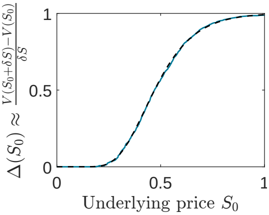
MC estimator allows stable differentiation (w.r.t. S_0) if path payoffs depend Lipschitz continuously on S_0 and on the random samples.

- Analogous result hold for other parameter, e.g., other greeks.

Consequence: For options with L.-continuous payoffs functions:

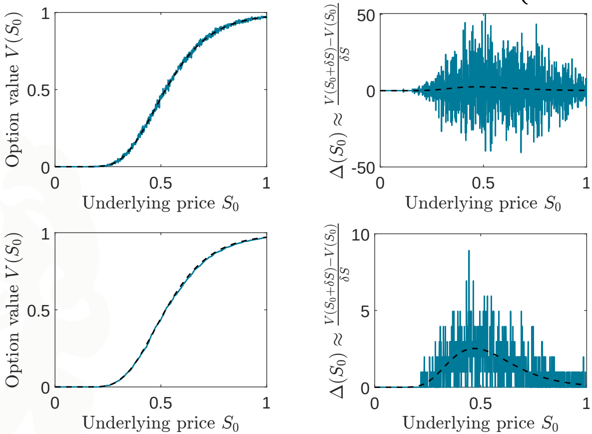
- Resetting random seed yields stable greeks





Example for discontinuous payoff function: Binary option

Binary “cash-or-nothing” call option: $V(S_T) = \begin{cases} 1 & \text{if } S_T > K, \\ 0 & \text{else.} \end{cases}$



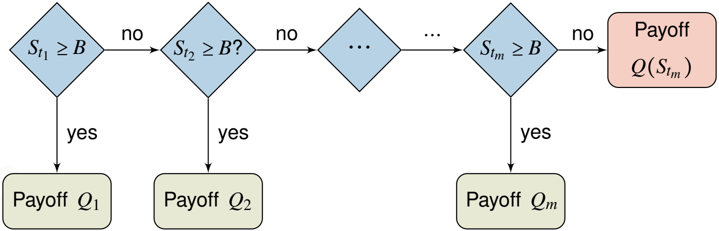
w/o resetting random seed

with resetting random seed

Unstable greeks with and w/o resetting random seed.

Stable greeks for discontinuous payoffs

Barrier option / Autocallable / Express Certificate



Discontinuous, path-dependent payoff:

$$V(S_{t_1}, \ldots, S_{t_m}) = \begin{cases} e^{-r(t_j - t_0)} Q_j & \text{if } S_{t_i} < B \leq S(t_j) \;\; \forall i < j, \\ e^{-r(t_m - t_0)} Q(S_{t_m}) & \text{if } S_{t_j} < B \qquad\qquad \forall j \in \{1, \ldots, m\}. \end{cases}$$

- Lots of variants, e.g. “down-and-out”, continuous barrier, multivariate options (on max/min of multiple underlyings), …

Stable greeks for autocallables

One-step-survival (Glassermann/Staum, Operations Research 2001)

- Enforce MC paths to not hit the barrier

  (conditional sampling, draw from truncated stochastic distribution)
- Correct for missing barrier hit
  - p: probability of path not hitting barrier in $j+1$-th step
  - Add $(1-p)e^{-r(t_{j+1}-t_0)}Q_{j+1}$ to path payoff
  - Reduce weight of all future path payoffs by p

Mathematical formulation (Alm/**H.**/Harrach/Keller, JCF 2013)

- Replace pathwise discontinuous payoff by (mathematically equivalent!) pathwise L.-continuous payoff

$\rightsquigarrow$ OSS-MC allows stable differentiation

Implementation

Naive MC ($\rightsquigarrow$ unstable greeks)

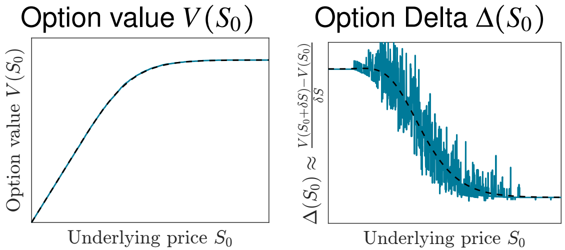
Reset random seed
for $n = 1,\ldots,N$ **do**
  for $j = 0,\ldots,m-1$ **do**
    $\tau := t_{j+1} - t_j$
    Sample $u_j \sim U(0,1)$
    $S_{j+1} := S_j e^{\left(r-\sigma^2/2\right)\tau+\sigma\sqrt{\tau}\Phi^{-1}(u_j)}$
    if $S_{j+1} \geq B$ **then**
      $P_n := e^{-r(t_{j+1}-t_0)} Q_{j+1}$; **break**
    else if $j+1 = m$ **then**
      $P_n := e^{-r(t_m-t_0)} Q(S_m)$
    end if
  end for
end for
return $V(S_0) \approx \frac{1}{N}\sum_{n=1}^{N} P_n$

OSS-MC ($\rightsquigarrow$ stable greeks)

Reset random seed
for $n = 1,\ldots,N$ **do**
  $P_n := 0, \quad L := 1$
  for $j = 0,\ldots,m-1$ **do**
    $\tau := t_{j+1} - t_j$
    Sample $u_j \sim U(0,1)$
    $p := \Phi\left(\frac{\ln(B/S_j)-(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right)$
    $S_{j+1} := S_j e^{\left(r-\sigma^2/2\right)\tau+\sigma\sqrt{\tau}\Phi^{-1}(pu_j)}$
    $P_n := P_n + (1-p)Le^{-r(t_{j+1}-t_0)} Q_{j+1}$
    $L := pL$
  end for
  $P_n := P_n + Le^{-r(t_m-t_0)} Q(S_m)$
end for
return $V(S_0) \approx \frac{1}{N}\sum_{n=1}^{N} P_n$

Easy-to-implement, replace if-clauses by probability weights

Example: Univariate Autocallable Option



Standard Monte Carlo (blue line)

- works well for pricing
- fails for calculating derivatives, e.g., $\Delta(S_0) = \frac{\partial}{\partial S} V(S_0)$

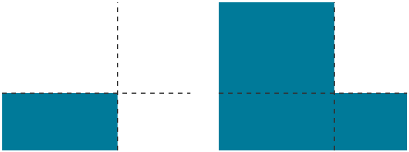
One-step-survival Monte Carlo (black line)

- works well for pricing and for calculating derivatives

Multivariate Case

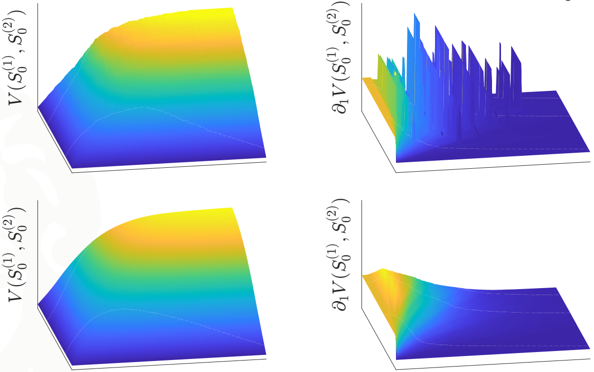
Multivariate case (Alm/**H.**/Harrach/Keller, JCF 2013):

- Barrier hit depends on max/min of multiple underlyings
- Requires smooth conditional sampling of survival zone
- Acception-rejection strategies not applicable (not smooth!)
- GHK importance sampling:
  - sample one dimension after the other (smooth)
  - correct for wrong prob. distribution by weight factors (smooth)
- For L-shaped survival zone (barrier on min. of underlyings): Additional rotation needed for continuous parametrization



Results for multivariate autocallable option

Option value $V(S_0)$ | Option Delta w.r.t. $S_0^{(1)}$



standard MC

OSS-GHK-MC

Summary

- Differentiation is intrinsically unstable
  - ⇝ Convergent pricing algorithm may still yield unstable greeks
- Stable differentiation possible if path payoffs depend L.-continously on parameter and random samples
  - ⇝ Stable greeks for discont. payoff possible by one-step-survival
  - ⇝ Multivariate cases can be treated by subsequent dimension sampling plus importance sampling

Extensions

- Ideas extend to pathwise sensitivity calculations, continuous barrier options, and parameter calibration problems

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References

- Alm/**H.**/Harrach/Keller: A Monte Carlo pricing algorithm for autocallables that allows for stable differentiation, *J. Comput. Finance* 2013
- Gerstner/**H.**/Roth: Monte Carlo pathwise sensitivities for barrier options, *JCF* 2020
- Gerstner/**H.**/Roth: Convergence of Milstein Brownian bridge Monte Carlo methods and stable Greeks calculation *arXiv:1906.11002*