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# Towards global convergence for inverse coefficient problems with finitely many measurements

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# Monotonicity and Convexity

# Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \subset \mathbb{R}^d \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves (1)}\}\,?$$

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Equivalent: Recover $\sigma$ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma):\ L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where $u$ solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

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# The Monotonicity Lemma

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**Lemma.** (First appearance: Kang/Seo/Sheen 1997, Ikehata 1998)

$$\int_{\partial\Omega} g(\Lambda(\sigma_1)-\Lambda(\sigma_2))g \, \mathrm{d}s \geq \int_\Omega (\sigma_2-\sigma_1)|\nabla u_{\sigma_2}^g|^2 \, \mathrm{d}x.$$

for all $\sigma_1, \sigma_2 \in L_+^\infty(\Omega)$, $g \in L_\diamond^2(\partial\Omega)$.

- Monotonicity w.r.t. Loewner order:

$$\sigma_1 \leq \sigma_2 \quad \Longrightarrow \quad \Lambda(\sigma_1) \geq \Lambda(\sigma_2)$$

⇝ Inclusion detection method (Tamburrino/Rubinacci 2002)

- Localized potentials:

$$\exists (g_k)_{k\in\mathbb{N}}: \int_{D_1} |\nabla u_\sigma^{g_k}|^2 \, \mathrm{d}x \to \infty, \quad \int_{D_2} |\nabla u_\sigma^{g_k}|^2 \, \mathrm{d}x \to 0.$$

⇝ Converse monotonicity holds for inclusion detection.

⇝ Monotonicity method yields exact shape (**H.**/Ullrich 2013).

# Monotonicity and Convexity

## Lemma.

$$\int_{\partial\Omega} g(\Lambda(\sigma_1)-\Lambda(\sigma_2))g \, \mathrm{d}s \geq \int_\Omega (\sigma_2-\sigma_1)|\nabla u_{\sigma_2}^g|^2 \, \mathrm{d}x$$
$$= \int_{\partial\Omega} g\Lambda'(\sigma_2)(\sigma_1-\sigma_2)g \, \mathrm{d}s.$$

for all $\sigma_1, \sigma_2 \in L_+^\infty(\Omega)$, $g \in L_\diamond^2(\partial\Omega)$.

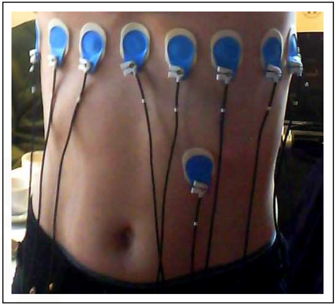
- ⇝ For all $\sigma_1, \sigma_2 \in L_+^\infty(\Omega)$: $\Lambda(\sigma_1)-\Lambda(\sigma_2) \geq \Lambda'(\sigma_2)(\sigma_1-\sigma_2)$.
- ⇝ Convexity: For all $\sigma_1, \sigma_2 \in L_+^\infty(\Omega)$, $t \in [0,1]$

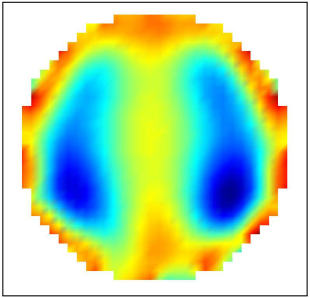
$$\Lambda((1-t)\sigma_1 + t\sigma_2) \leq (1-t)\Lambda(\sigma_1) + t\Lambda(\sigma_2).$$

*The "monotonicity lemma" also implies convexity.*

# EIT with finitely many measurements

# Electrical impedance tomography (EIT)



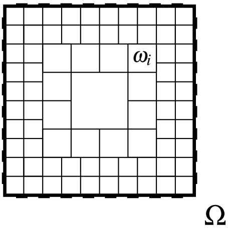


- Apply electric currents on subject's boundary
- Measure necessary voltages

⇝ Reconstruct conductivity inside subject

# EIT in practice

- Finitely many unknowns, $\sigma$ pcw. const. on given resolution $\Omega = \bigcup_{i=1}^{n} \omega_i$
- Finitely many measurements
$$\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \,\mathrm{d}s$$
for given currents $g_1, \ldots, g_m \in L^2_\diamond(\partial\Omega)$

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## Finite-dimensional inverse problem: Determine

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix} \in \mathbb{R}^n_+ \quad \text{from } F(\sigma) = \left( \int_{\partial\Omega} g_j \Lambda(\sigma) g_k \,\mathrm{d}s \right)_{j,k=1}^{m} \in \mathbb{R}^{m\times m}.$$

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# Mathematical challenges for practical EIT

Inverse problem: Determine $\sigma \in \mathbb{R}_+^n$ from $Y = F(\sigma) \in \mathbb{R}^{m\times m}$.

For a fixed desired resolution:

- How many measurements uniquely determine $\sigma$?
- Stability / error estimates for noisy data $Y^\delta \approx F(\sigma)$?
- Numerical algorithm to determine $\sigma \in \mathbb{R}_+^n$ from $Y^\delta \approx F(\sigma)$?
- Global/local convergence of algorithm?

*Can we utilize the monotonicity and convexity properties?*

# Simple example: EIT with 2 unknowns & 6 bndry. currents

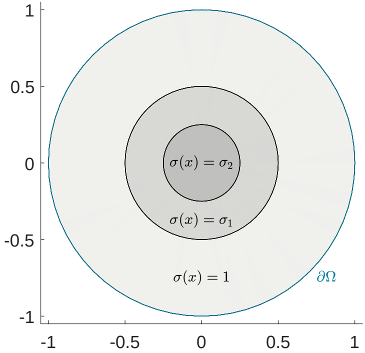
$\Omega$: unit circle

$$F: \mathbb{R}^2_+ \to \mathbb{R}^{6\times 6}$$

$$F\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} := \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k\right)_{j,k=1}^{6}$$

with trigonometric currents

$$\{g_1,\ldots,g_6\} = \{\sin(\varphi),\ldots,\cos(3\varphi)\}$$

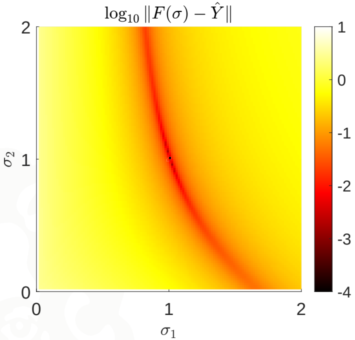


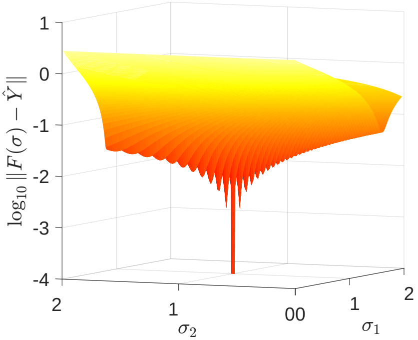

Inverse problem: Reconstruct $\hat{\sigma} \in \mathbb{R}^2_+$ from $\hat{Y} = F(\hat{\sigma}) \in \mathbb{R}^{6\times 6}$

Natural approach: Least squares data fitting

$$\texttt{minimize} \quad \|F(\sigma) - \hat{Y}\|_{\mathrm{F}}^2 \qquad \textit{(+ Regularization)}$$

# Problem of local minima



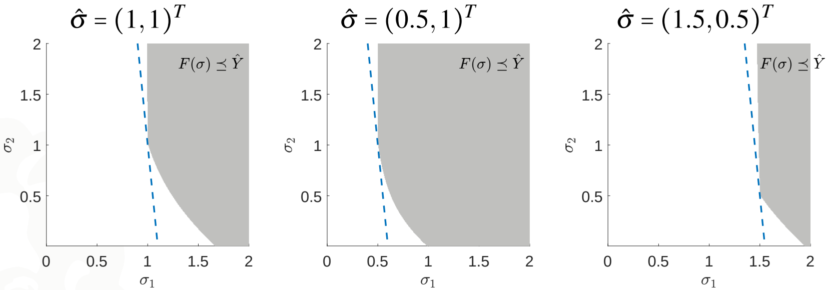

Numerical results indicate

- $\hat{Y} = F(\hat{\sigma})$ uniquely determines $\hat{\sigma}\ldots$
- . . . but residuum is highly non-convex, many local minima

Are globally convergent algorithms impossible?

# Bold guess

## Inverse problem: Reconstruct $\hat{\sigma} \in \mathbb{R}^2_+$ from $\hat{Y} = F(\hat{\sigma}) \in \mathbb{R}^{6\times 6}$



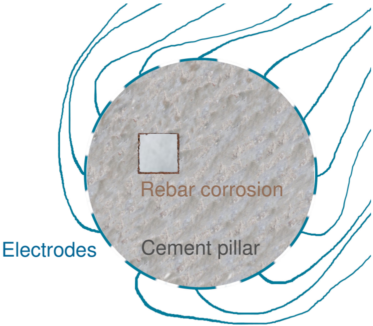

## Bold conjecture.

$\hat{\sigma}$ is the lower left corner of the convex set $F(\sigma) \leq \hat{Y}$.

*(Conjecture yet unproven for EIT. Next slides show proven variant for Robin problem.)*

# An inverse Robin coefficient problem

## EIT for corrosion detection




Non-destructive EIT-based corrosion detection:

- Apply electric currents on outer boundary $\partial\Omega$
- Measure necessary voltages

$\rightsquigarrow$ Detect corrosion on inner boundary $\Gamma = \partial D$

## Idealized mathematical model: Robin PDE

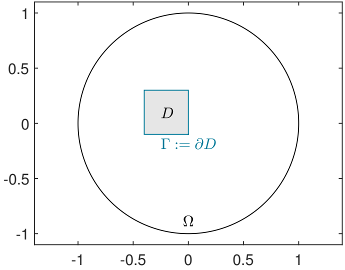
Electric potential $u: \Omega \to \mathbb{R}$ solves

(1) $\Delta u = 0$ in $\Omega \setminus \Gamma$,

(2) $\partial_\nu u|_{\partial\Omega} = g$ on $\partial\Omega$,

(3) $[\![u]\!]_\Gamma = 0$ on $\Gamma$,

(4) $[\![\partial_\nu u]\!]_\Gamma = \sigma u$ on $\Gamma$




Inverse Problem: Recover $\sigma$ from Neumann-to-Dirichlet-Operator

$$\Lambda(\sigma): L^2(\partial\Omega) \to L^2(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where $u$ solves Robin PDE (1)–(4).

## Finitely many measurements and unknowns

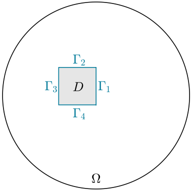
- Finitely many measurements:

$$\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, \mathrm{d}s \quad \text{for finitely many } g_1, \dots, g_m$$

- Finite desired resolution:

$$\sigma = \sum_{j=1}^{n} \sigma_j \chi_{\Gamma_j} \quad \text{with } \sigma_j \in \mathbb{R},\ j = 1, \dots, n$$

*with partition* $\Gamma = \bigcup_{j=1}^{n} \Gamma_j$




- A-priori bounds: $\sigma := (\sigma_1, \dots, \sigma_n)^T \in [a,b]^n$, $b > a > 0$ known

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Finite-dimensional non-linear inverse problem: Determine

$$\sigma = (\sigma_j)_{j=1}^{n} \in [a,b]^n \quad \text{from} \quad F(\sigma) := \left( \int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, \mathrm{d}s \right)_{j,k=1}^{m} \in \mathbb{R}^{m\times m}$$

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## Main result 1/3

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**Theorem.** (**H.**, Optim. Lett. 2021)

If sufficiently many measurements are taken, then

- $\hat{Y} := F(\hat{\sigma}) \in \mathbb{R}^{m \times m}$ uniquely determines $\hat{\sigma} \in [a,b]^n$.
- $\hat{\sigma}$ is the unique solution of

$$\texttt{minimize} \quad \|\sigma\|_1 = \sum_{j=1}^{n} \sigma_j \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \leq \hat{Y}.$$

- The constraint set $\sigma \in [a,b]^n,\ F(\sigma) \leq \hat{Y}$ is convex.

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⇝ $\hat{\sigma}$ is the lower left corner of the convex constraint set

⇝ Problem can be solved by convex semidefinite programming

*Global convergence is feasible.*

(**H.**, Numer. Math. 2020: Global Newton convergence for this Robin problem)

## Main result 2/3

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**Theorem.** (**H.**, Optim. Lett. 2021)

- Suff. many measurements are taken if $\lambda_{\text{max}}(F'(z_{j,k})d_j) > 0$ for
$$z_{j,k} := \frac{a}{2}e'_j + \left(a + k\frac{a}{4(n-1)}\right)e_j \in \mathbb{R}^n_+, \quad d_j := \frac{2b-a}{a}(n-1)e'_j - \frac{1}{2}e_j \in \mathbb{R}^n,$$
with $j = 1, \ldots, n, \quad k = 1, \ldots, \lceil \frac{4(n-1)b}{a} \rceil - 4n + 5.$
- This criterion is fulfilled if $(g_j)_{j=1}^{\infty}$ has dense span in $L^2(\partial\Omega)$, and sufficiently many $g_j$ are used.

*($e_j \in \mathbb{R}^n$: $j$-th unit vector, $e'_j := \mathbb{1} - e_j \in \mathbb{R}^n$: negated $j$-th unit vector)*

⇝ Explicit, easy-to-check criterion whether a desired resolution can be achieved with a certain number of measurements

*Achievable resolution can be characterized.*

# Main result 3/3

**Theorem.** (**H.**, Optim. Lett. 2021)

- Let the criterion hold with lower bound $\lambda > 0$.
- Let $\delta > 0$, and $Y^\delta \in \mathbb{R}^{m\times m}$ be symmetric with $\|\hat{Y} - Y^\delta\|_2 \leq \delta$.

Then there exist solutions of

$$\texttt{minimize} \quad \|\sigma\|_1 = \sum_{j=1}^{n} \sigma_j \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \preceq Y^\delta + \delta I.$$

and every such minimum $\sigma^\delta$ fulfills

$$\|\hat{\sigma} - \sigma^\delta\|_\infty \leq \frac{2\delta(n-1)}{\lambda}$$

*Explicit error estimates, convergence for $\delta \to 0$.*

# Proof ingredients & possible generalizations

- Monotonicity & Convexity: $F : \mathbb{R}^n_+ \to \mathbb{S}_m \subset \mathbb{R}^{m\times m}$ fulfills

$$F'(\sigma)d \leq 0 \qquad \text{for all } \sigma \in \mathbb{R}^n_+,\ 0 \leq d \in \mathbb{R}^n$$
$$F(\tau) - F(\sigma) \geq F'(\sigma)(\tau - \sigma) \qquad \text{for all } \sigma, \tau \in \mathbb{R}^n_+$$

$\rightsquigarrow$ holds for general elliptic PDEs (**H.**, Jahresber. DMV, 2021)

- Localized potentials: For any $C > 0$, there exist currents $g$ s.t.

$$g^T(F'(\sigma)(e_j - Ce_j'))g = \int_{\Gamma_j} |\nabla u|^2 \, \mathrm{d}x - C\int_{\Gamma\setminus\Gamma_j} |\nabla u|^2 > 0$$

$\Longrightarrow$ $\lambda_{\text{max}}(F'(z)(e_j - Ce_j')) > 0$ for suff. many measurem.

$\rightsquigarrow$ holds for many elliptic problems, but in more complicated form

## Conclusions

### For elliptic coefficient inverse problems

- least-squares residuum functionals may be highly non-convex
- local minima are usually useless

### Possible remedy

- utilize monotonicity & convexity with respect to Loewner order
- utilize localized potentials to control directional derivatives

### For an inverse Robin coefficient problem we can obtain

- equivalent reformulation as convex semidefinite program
- globally convergent solution algorithms
- explicit characterizations of achievable resolution
- explicit error estimates for noisy data