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Global convergence and stable invertibility for a ~~Robin transmission~~ Calderón problem with finitely many measurements

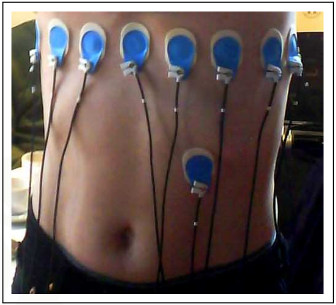
Bastian Harrach

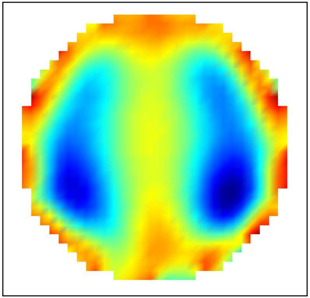
http://numerical.solutions

Institute of Mathematics, Goethe University Frankfurt, Germany

10th International Conference
Inverse Problems: Modelling and Simulation (IPMS)
Malta, May 22–28, 2022.

Electrical impedance tomography (EIT)





- Apply electric currents on subject's boundary
- Measure necessary voltages

⇝ Reconstruct conductivity inside subject

Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \subset \mathbb{R}^d \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves (1)}\}\,?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma):\ L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Challenges in idealized EIT

Mathematical idealization of EIT $\rightsquigarrow$ Calderón problem

- infinitely many unknowns $\sigma \in L^\infty_+(\Omega)$
- infinitely many measurements $\Lambda(\sigma) \in \mathcal{L}(L^2_\diamond(\partial\Omega))$
- nonlinear forward map $\sigma \mapsto \Lambda(\sigma)$

Mathematical challenges (and a very incomplete literature list)

- Uniqueness? Does $\Lambda(\sigma)$ determine σ?
  (Calderón 1980, Astala/Päivärinta '06, Krupchyk/Uhlmann '16, Caro/Rogers '16, ...)
- Stability? $\Lambda^{-1}: \Lambda(\sigma) \mapsto \sigma$ continuous?
  (Alessandrini/Vessella '05, Beretta/Francini '11, Rüland/Sincich '19, Alberti/Santacesaria '19, ...)
- Convergence (local/global)? How to determine σ from $\Lambda(\sigma)$?
  (Local convergence/TCC: *Lechleiter/Rieder '08, Kindermann '21*, Global convergence: *Klibanov/Zhang '19*)

Consequences for practical EIT?

EIT in practice

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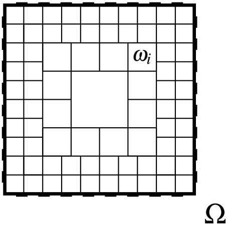
- Finitely many unknowns, σ pcw. const. on given resolution $\Omega = \bigcup_{i=1}^{n} \omega_i$
- Finitely many measurements

$$\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, \mathrm{d}s$$

for given currents $g_1, \ldots, g_m \in L^2_\diamond(\partial\Omega)$



Finite-dimensional inverse problem: Determine

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix} \in \mathbb{R}^n_+ \quad \text{from } F(\sigma) = \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, \mathrm{d}s \right)_{j,k=1}^{m} \in \mathbb{S}_m \subset \mathbb{R}^{m\times m}.$$

Mathematical challenges for practical EIT

Inverse problem: Determine $\sigma \in \mathbb{R}^n_+$ from $Y = F(\sigma) \in \mathbb{R}^{m\times m}$.

For a fixed desired resolution:

- How many measurements uniquely determine σ?
- Stability / error estimates for noisy data $Y^\delta \approx F(\sigma)$?
- Numerical algorithm to determine $\sigma \in \mathbb{R}^n_+$ from $Y^\delta \approx F(\sigma)$?
- Global/local convergence of algorithm?

Next slides: The problem of local convergence and a convex reformulation

Simple example: EIT with 2 unknowns & 6 bndry. currents

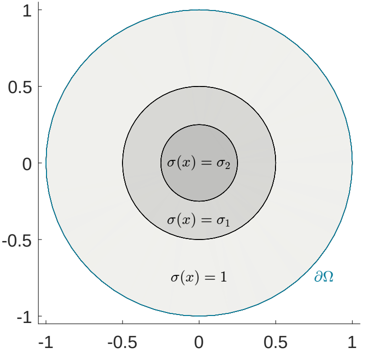
Ω: unit circle

$$F:\ \mathbb{R}^2_+ \to \mathbb{R}^{6\times 6}$$

$$F\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} := \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k\right)_{j,k=1}^{6}$$

with trigonometric currents

$$\{g_1,\ldots,g_6\} = \{\sin(\varphi),\ldots,\cos(3\varphi)\}$$

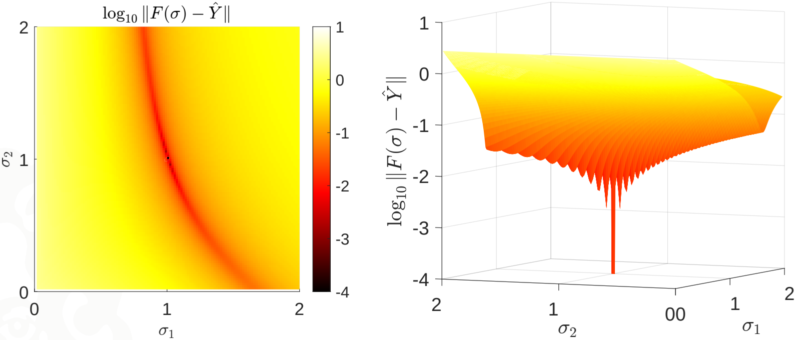


Inverse problem: Reconstruct $\hat\sigma \in \mathbb{R}^2_+$ from $\hat Y = F(\hat\sigma) \in \mathbb{S}_6 \subset \mathbb{R}^{6\times 6}$

Natural approach: Least squares data fitting

$$\texttt{minimize} \quad \|F(\sigma) - \hat Y\|_{\mathsf{F}}^2 \qquad \textit{(+ Regularization)}$$

Problem of local minima



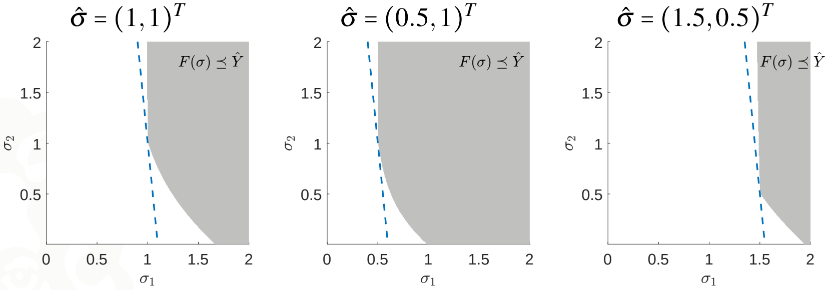
Numerical results indicate

- $\hat{Y} = F(\hat{\sigma})$ uniquely determines $\hat{\sigma}$...
- ...but residuum is highly non-convex, many local minima

Are globally convergent algorithms impossible?

Conjecture

Inverse problem: Reconstruct $\hat{\sigma} \in \mathbb{R}^2_+$ from $\hat{Y} = F(\hat{\sigma}) \in \mathbb{R}^{6\times 6}$

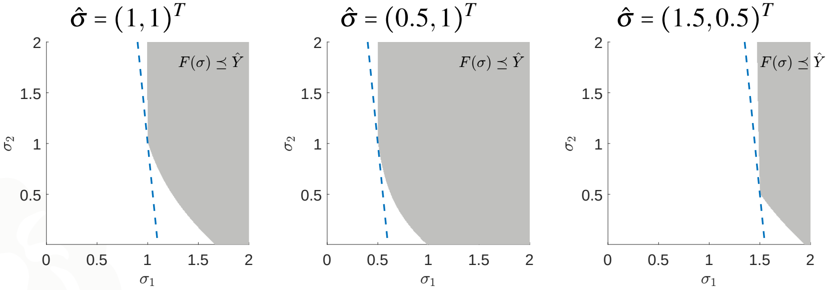


Conjecture.

$\hat{\sigma}$ is the lower left corner of the convex set $F(\sigma) \preceq \hat{Y}$.

("$\preceq$": Loewner / semidefiniteness order)

Mathematical formulation



Conjecture. $\exists c \in \mathbb{R}^n$ so that true solution $\hat\sigma$ minimizes

$$c^T \sigma = \sum_{i=1}^n c_i \sigma_i \to \text{min!} \quad \text{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \leq \hat Y.$$

For a similar but simpler Robin problem:

- Conjecture holds with $c = \mathbb{1}$ *(**H.**, Optim. Lett. 2021)*
- Global Newton convergence is possible *(**H.**, Numer. Math. 2021)*

Convex reformulation for EIT

Theorem. (**H.**, arxiv:2203.16779)
If sufficiently many measurements are taken, then:

- EIT forward mapping $F: [a,b]^n \to \mathbb{S}_m \subset \mathbb{R}^{m\times m}$ is injective.
- Derivative $F'(\sigma)$ is injective for all $\sigma \in [a,b]^n$.
- There exists $c \in \mathbb{R}^n_+$ so that for all $\hat{\sigma} \in [a,b]^n$, $\hat{Y} = \Lambda(\hat{\sigma})$:

  $\hat{\sigma}$ is the unique solution of the convex problem

$$\texttt{minimize} \quad c^T\sigma = \sum_{i=1}^{n} c_i \sigma_i \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \preceq \hat{Y}.$$

The Calderón problem with finitely many unknowns is equivalent to convex semidefinite optimization

Stability and error estimates

Theorem (continued). (**H.**, arxiv:2203.16779)

There exists $\lambda > 0$ so that

- for all $\hat{\sigma} \in [a,b]^n$, $\hat{Y} := \Lambda(\hat{\sigma})$,
- and all $\delta > 0$, $Y^\delta \in \mathbb{S}_m \subset \mathbb{R}^{m\times m}$, with $\|Y^\delta - \hat{Y}\| \leq \delta$,

the convex semidefinite optimization problem

$$\texttt{minimize} \quad c^T\sigma = \sum_{i=1}^{n} c_i\sigma_i \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \leq Y^\delta + \delta I.$$

possesses a minimizer σ^δ. Every such minimizer fulfills

$$\|\sigma^\delta - \hat{\sigma}\|_{c,\infty} \leq \frac{n-1}{\lambda}\delta.$$

($\|\cdot\|_{c,\infty}$: c-weighted maximum norm)

Error estimates for noisy data $Y^\delta \approx \hat{Y}$ also hold.

Conclusions

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For elliptic coefficient inverse problems

- least-squares residuum functionals may be highly non-convex
- local minima are usually useless

Possible remedy

- utilize monotonicity & convexity with respect to Loewner order
- utilize localized potentials to control directional derivatives

Equivalent convex reformulations are possible

- globally convergent solution algorithms are possible
- error estimates for noisy data are possible
- For simple Robin problem
  - explicit characterizations of achievable resolution
  - explicit error estimates for noisy data