Towards global convergence for inverse coefficient problems

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Monotonicity and Convexity

Calderón problem

Can we recover $\sigma \in L_+^\infty(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \subset \mathbb{R}^d \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves (1)}\}\,?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma) : L_\diamond^2(\partial\Omega) \to L_\diamond^2(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

The Monotonicity Lemma

Lemma. (First appearance: Kang/Seo/Sheen 1997, Ikehata 1998)

$$\int_{\partial\Omega} g(\Lambda(\sigma_1) - \Lambda(\sigma_2))g \, \mathrm{d}s \geq \int_{\Omega} (\sigma_2 - \sigma_1) |\nabla u_{\sigma_2}^g|^2 \, \mathrm{d}x.$$

for all $\sigma_1, \sigma_2 \in L^\infty_+(\Omega)$, $g \in L^2_\diamond(\partial\Omega)$.

- Monotonicity w.r.t. Loewner order:

$$\sigma_1 \leq \sigma_2 \quad \Longrightarrow \quad \Lambda(\sigma_1) \geq \Lambda(\sigma_2)$$

  ⇝ Inclusion detection method (Tamburrino/Rubinacci 2002)

- Localized potentials:

$$\exists (g_k)_{k\in\mathbb{N}} : \int_{D_1} |\nabla u_\sigma^{g_k}|^2 \, \mathrm{d}x \to \infty, \quad \int_{D_2} |\nabla u_\sigma^{g_k}|^2 \, \mathrm{d}x \to 0.$$

  ⇝ Converse monotonicity holds for inclusion detection.

  ⇝ Monotonicity method yields exact shape (**H.**/Ullrich 2013).

Monotonicity and Convexity

Lemma.

$$\int_{\partial\Omega} g(\Lambda(\sigma_1)-\Lambda(\sigma_2))g \, \mathrm{d}s \geq \int_\Omega (\sigma_2-\sigma_1)|\nabla u_{\sigma_2}^g|^2 \, \mathrm{d}x$$
$$= \int_{\partial\Omega} g\Lambda'(\sigma_2)(\sigma_1-\sigma_2)g \, \mathrm{d}s.$$

for all $\sigma_1, \sigma_2 \in L_+^\infty(\Omega)$, $g \in L_\diamond^2(\partial\Omega)$.

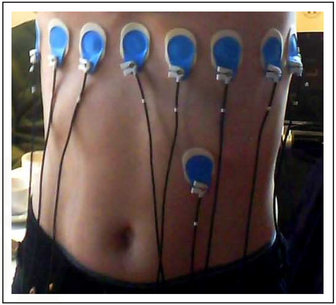
- ⤳ For all $\sigma_1, \sigma_2 \in L_+^\infty(\Omega)$: $\Lambda(\sigma_1)-\Lambda(\sigma_2) \geq \Lambda'(\sigma_2)(\sigma_1-\sigma_2)$.
- ⤳ Convexity: For all $\sigma_1, \sigma_2 \in L_+^\infty(\Omega)$, $t \in [0,1]$

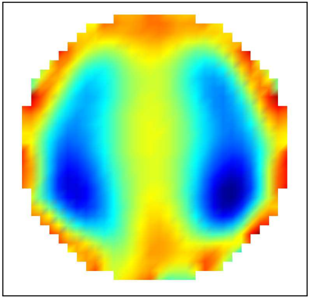
$$\Lambda((1-t)\sigma_1 + t\sigma_2) \leq (1-t)\Lambda(\sigma_1) + t\Lambda(\sigma_2).$$

The "monotonicity lemma" also implies convexity.

EIT with finitely many measurements

Electrical impedance tomography (EIT)





- Apply electric currents on subject's boundary
- Measure necessary voltages

⤳ Reconstruct conductivity inside subject

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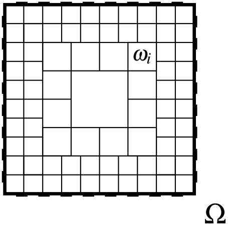
EIT in practice

- Finitely many unknowns, σ pcw. const. on given resolution $\Omega = \bigcup_{i=1}^{n} \omega_i$
- Finitely many measurements

$$\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, \mathrm{d}s$$

for given currents $g_1, \ldots, g_m \in L^2_\diamond(\partial\Omega)$



Finite-dimensional inverse problem: Determine

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix} \in \mathbb{R}^n_+ \quad \text{from } F(\sigma) = \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, \mathrm{d}s \right)_{j,k=1}^{m} \in \mathbb{R}^{m \times m}.$$

Mathematical challenges for practical EIT

Inverse problem: Determine $\sigma \in \mathbb{R}_+^n$ from $Y = F(\sigma) \in \mathbb{R}^{m \times m}$.

For a fixed desired resolution:

- How many measurements uniquely determine σ?
- Stability / error estimates for noisy data $Y^\delta \approx F(\sigma)$?
- Numerical algorithm to determine $\sigma \in \mathbb{R}_+^n$ from $Y^\delta \approx F(\sigma)$?
- Global/local convergence of algorithm?

Can we utilize the monotonicity and convexity properties?

Simple example: EIT with 2 unknowns & 6 bndry. currents

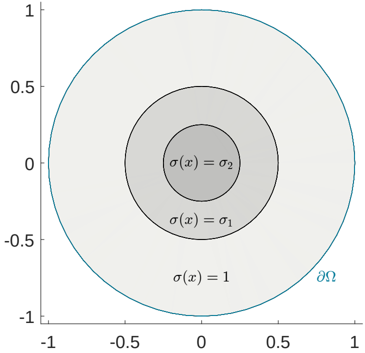
Ω: unit circle

$$F: \mathbb{R}_+^2 \to \mathbb{R}^{6\times 6}$$

$$F\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} := \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k\right)_{j,k=1}^{6}$$

with trigonometric currents

$$\{g_1,\ldots,g_6\} = \{\sin(\varphi),\ldots,\cos(3\varphi)\}$$

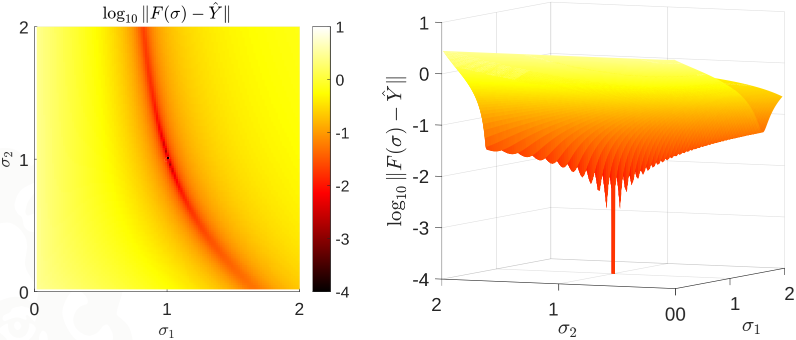


Inverse problem: Reconstruct $\hat{\sigma} \in \mathbb{R}_+^2$ from $\hat{Y} = F(\hat{\sigma}) \in \mathbb{R}^{6\times 6}$

Natural approach: Least squares data fitting

$$\texttt{minimize} \quad \|F(\sigma) - \hat{Y}\|_F^2 \qquad \textit{(+ Regularization)}$$

Problem of local minima



Numerical results indicate

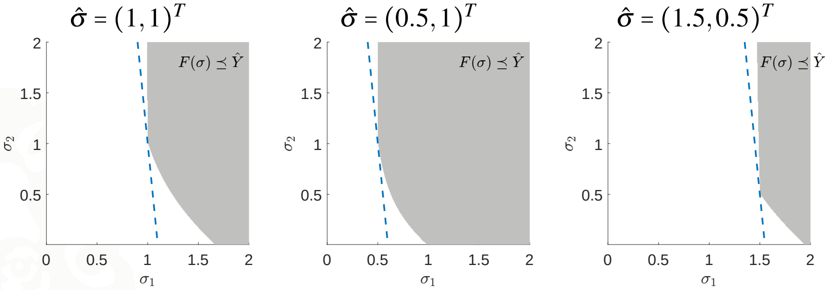
- $\hat{Y} = F(\hat{\sigma})$ uniquely determines $\hat{\sigma}$...
- ...but residuum is highly non-convex, many local minima

Are globally convergent algorithms impossible?

Towards global convergence

A bold conjecture

Inverse problem: Reconstruct $\hat{\sigma} \in \mathbb{R}^2_+$ from $\hat{Y} = F(\hat{\sigma}) \in \mathbb{R}^{6\times 6}$

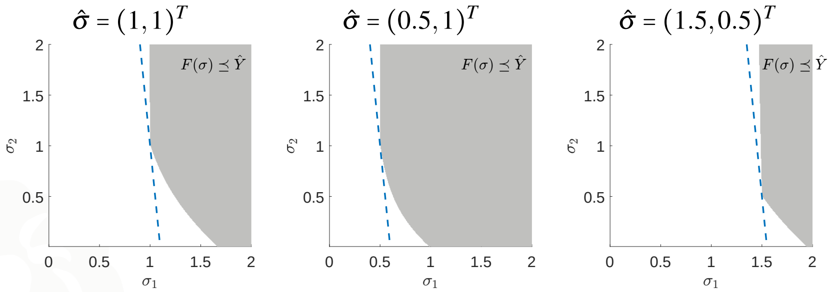


Conjecture.

$\hat{\sigma}$ is the lower left corner of the convex set $F(\sigma) \leq \hat{Y}$.

("≤": Loewner / semidefiniteness order)

Mathematical formulation



Conjecture. $\exists c \in \mathbb{R}^n$ so that true solution $\hat{\sigma}$ minimizes

$$c^T \sigma = \sum_{i=1}^{n} c_i \sigma_i \to \text{min!} \quad \text{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \leq \hat{Y}.$$

For a similar but simpler Robin problem:

- Conjecture holds with $c = \mathbb{1}$ *(**H.**, Optim. Lett. 2021)*
- Global Newton convergence is possible *(**H.**, Numer. Math. 2021)*

Convex reformulation for EIT

Theorem. (**H.**, arxiv:2203.16779)

If sufficiently many measurements are taken, then:

- EIT forward mapping $F:\ [a,b]^n \to \mathbb{S}_m \subset \mathbb{R}^{m\times m}$ is injective.
- Derivative $F'(\sigma)$ is injective for all $\sigma \in [a,b]^n$.
- There exists $c \in \mathbb{R}^n_+$ so that for all $\hat{\sigma} \in [a,b]^n$, $\hat{Y} = \Lambda(\hat{\sigma})$:

  $\hat{\sigma}$ is the unique solution of the convex problem

$$\texttt{minimize} \quad c^T\sigma = \sum_{i=1}^{n} c_i\sigma_i \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \preceq \hat{Y}.$$

The Calderón problem with finitely many unknowns is equivalent to convex semidefinite optimization

Stability and error estimates

Theorem (continued). (**H.**, arxiv:2203.16779)

There exists $\lambda > 0$ so that

- for all $\hat{\sigma} \in [a,b]^n$, and $\hat{Y} := \Lambda(\hat{\sigma})$,
- and all $\delta > 0$, and $Y^\delta \in \mathbb{S}_m \subset \mathbb{R}^{m\times m}$, with $\|Y^\delta - \hat{Y}\| \leq \delta$,

the convex semidefinite optimization problem

$$\texttt{minimize} \quad c^T \sigma = \sum_{i=1}^n c_i \sigma_i \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \leq Y^\delta + \delta I.$$

possesses a minimizer σ^δ. Every such minimizer fulfills

$$\|\sigma^\delta - \hat{\sigma}\|_{c,\infty} \leq \frac{n-1}{\lambda}\delta.$$

($\|\cdot\|_{c,\infty}$: c-weighted maximum norm)

Error estimates for noisy data $Y^\delta \approx \hat{Y}$ also hold.

Conclusions 1/2

For elliptic coefficient inverse problems

- least-squares residuum functionals may be highly non-convex
- local minima are usually useless

Possible remedy

- utilize monotonicity & convexity with respect to Loewner order
- utilize localized potentials to control directional derivatives

Equivalent convex reformulations are possible

- globally convergent solution algorithms are possible
- error estimates for noisy data are possible
- For similar but simpler Robin problem
  - explicit characterizations of achievable resolution
  - explicit error estimates for noisy data

Conclusions 2/2 *(now getting very subjective)*

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Some future challenges in inverse problems in PDEs:

We need to progress and extend...

FROM: Uniqueness results for infinite-dimensional DtN/NtD
TO: Resolution attainable from finitely many measurements

FROM: Stability results
TO: Error estimates (with computable constants)

FROM: Local convergence and non-convex residuum functionals
TO: Global convergence and convex functionals

Loewner monotonicity & convexity can help with these challenges.