Uniqueness and global convergence for inverse coefficient problems with finitely many measurements

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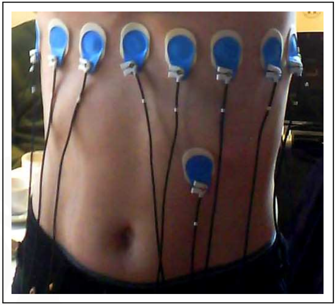
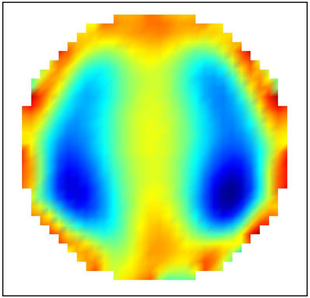
2nd Alps-Adriatic Inverse Problems Workshop 2021
Chemnitz Inverse Problems Symposium on tour
University of Klagenfurt, Austria, September 22–24, 2021.

EIT with finitely many measurements

Electrical impedance tomography (EIT)





- Apply electric currents on subject's boundary
- Measure necessary voltages

⇝ Reconstruct conductivity inside subject

Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \subset \mathbb{R}^d \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves } (1)\} \, ?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma): \; L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Challenges in idealized EIT

Mathematical idealization of EIT $\rightsquigarrow$ Calderón problem

- infinitely many unknowns $\sigma \in L^\infty_+(\Omega)$
- infinitely many measurements $\Lambda(\sigma) \in \mathcal{L}(L^2_\diamond(\partial\Omega))$
- nonlinear forward map $\sigma \mapsto \Lambda(\sigma)$

Mathematical challenges

- Uniqueness? Does $\Lambda(\sigma)$ determine σ?
- Stability? $\Lambda^{-1}: \Lambda(\sigma) \mapsto \sigma$ continuous?
- Convergence (local/global)? How to determine σ from $\Lambda(\sigma)$?

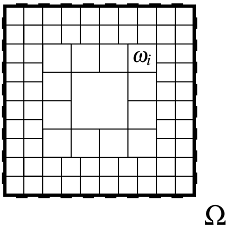
Consequences for practical EIT?

EIT in practice

- Finitely many unknowns, σ pcw. const. on given resolution $\Omega = \bigcup_{i=1}^{n} \omega_i$
- Finitely many measurements

$$\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, \mathrm{d}s$$

for given currents $g_1, \ldots, g_m \in L^2_\diamond(\partial\Omega)$



Finite-dimensional inverse problem: Determine

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix} \in \mathbb{R}^n_+ \quad \text{from } F(\sigma) = \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \, \mathrm{d}s \right)_{j,k=1}^{m} \in \mathbb{R}^{m\times m}.$$

Mathematical challenges for practical EIT

Inverse problem: Determine $\sigma \in \mathbb{R}^n_+$ from $Y = F(\sigma) \in \mathbb{R}^{m \times m}$.

For a fixed desired resolution:

- How many measurements uniquely determine σ?
- Stability / error estimates for noisy data $Y^\delta \approx F(\sigma)$?
- Numerical algorithm to determine $\sigma \in \mathbb{R}^n_+$ from $Y^\delta \approx F(\sigma)$?
- Global/local convergence of algorithm?

This talk: The problem of local convergence, a bold guess, and answers for a Robin problem (similar to but simpler than EIT)

The problem of local minima and a bold guess

Simple example: EIT with 2 unknowns & 6 bndry. currents

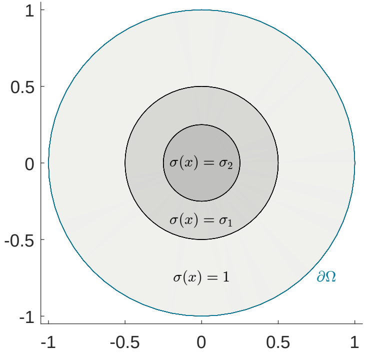
Ω: unit circle

$$F: \mathbb{R}^2_+ \to \mathbb{R}^{6\times 6}$$

$$F\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} := \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k\right)_{j,k=1}^{6}$$

with trigonometric currents

$$\{g_1,\ldots,g_6\} = \{\sin(\varphi),\ldots,\cos(3\varphi)\}$$

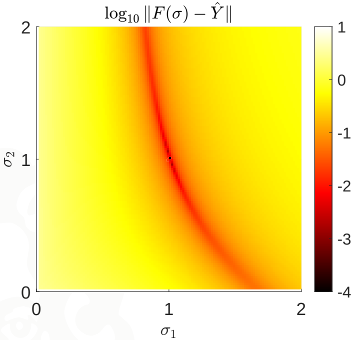


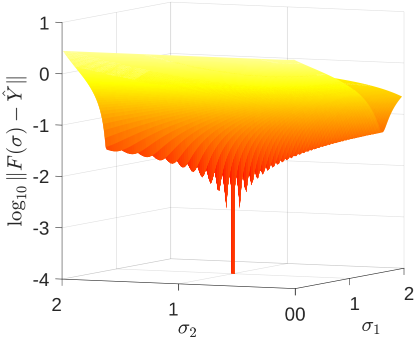
Inverse problem: Reconstruct $\hat{\sigma} \in \mathbb{R}^2_+$ from $\hat{Y} = F(\hat{\sigma}) \in \mathbb{R}^{6\times 6}$

Natural approach: Least squares data fitting

$$\texttt{minimize} \quad \|F(\sigma) - \hat{Y}\|_{\mathsf{F}}^2 \qquad \textit{(+ Regularization)}$$

Problem of local minima





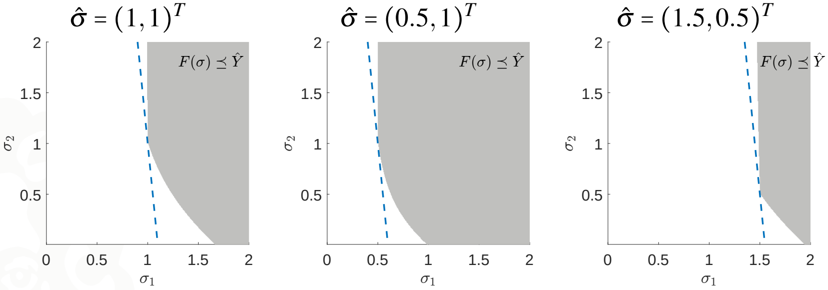
Numerical results indicate

- $\hat{Y} = F(\hat{\sigma})$ uniquely determines $\hat{\sigma}$...
- ...but residuum is highly non-convex, many local minima

Are globally convergent algorithms impossible?

Bold guess

Inverse problem: Reconstruct $\hat\sigma \in \mathbb{R}^2_+$ from $\hat Y = F(\hat\sigma) \in \mathbb{R}^{6\times 6}$



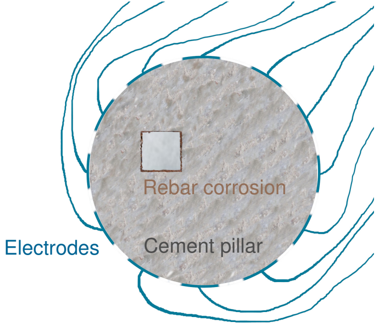
Bold conjecture.

$\hat\sigma$ is the lower left corner of the convex set $F(\sigma) \le \hat Y$.

("≤": Loewner / semidefiniteness order)

An inverse Robin coefficient problem

EIT for corrosion detection



Non-destructive EIT-based corrosion detection:

- Apply electric currents on outer boundary $\partial\Omega$
- Measure necessary voltages

$\leadsto$ Detect corrosion on inner boundary $\Gamma = \partial D$

Idealized mathematical model: Robin PDE

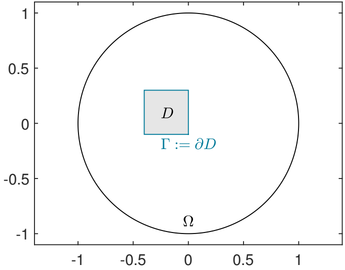
Electric potential $u: \Omega \to \mathbb{R}$ solves

(1) $\Delta u = 0$ in $\Omega \setminus \Gamma$,

(2) $\partial_\nu u|_{\partial\Omega} = g$ on $\partial\Omega$,

(3) $[\![u]\!]_\Gamma = 0$ on Γ,

(4) $[\![\partial_\nu u]\!]_\Gamma = \sigma u$ on Γ



Inverse Problem: Recover σ from Neumann-to-Dirichlet-Operator

$$\Lambda(\sigma): L^2(\partial\Omega) \to L^2(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves Robin PDE (1)–(4).

Finitely many measurements and unknowns

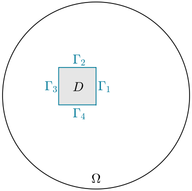
- Finitely many measurements:

$$\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \,\mathrm{d}s \quad \text{for finitely many } g_1, \ldots, g_m$$

- Finite desired resolution:

$$\sigma = \sum_{j=1}^{n} \sigma_j \chi_{\Gamma_j} \quad \text{with } \sigma_j \in \mathbb{R},\ j = 1, \ldots, n$$

with partition $\Gamma = \bigcup_{j=1}^{n} \Gamma_j$



- A-priori bounds: $\sigma := (\sigma_1, \ldots, \sigma_n)^T \in [a,b]^n$, $b > a > 0$ known

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Finite-dimensional non-linear inverse problem: Determine

$$\sigma = (\sigma_j)_{j=1}^{n} \in [a,b]^n \quad \text{from} \quad F(\sigma) := \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \,\mathrm{d}s \right)_{j,k=1}^{m} \in \mathbb{R}^{m\times m}$$

Main result 1/3

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Theorem. (**H.**, Optim. Lett. 2021)
If sufficiently many measurements are taken, then

- $\hat{Y} := F(\hat{\sigma}) \in \mathbb{R}^{m\times m}$ uniquely determines $\hat{\sigma} \in [a,b]^n$.
- $\hat{\sigma}$ is the unique solution of

$$\texttt{minimize} \quad \|\sigma\|_1 = \sum_{j=1}^{n} \sigma_j \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \leq \hat{Y}.$$

- The constraint set $\sigma \in [a,b]^n$, $F(\sigma) \leq \hat{Y}$ is convex.

$\rightsquigarrow$ $\hat{\sigma}$ is the lower left corner of the convex constraint set

$\rightsquigarrow$ Problem can be solved by convex semidefinite programming

Global convergence is feasible.

(**H.**, Numer. Math. 2020: Global Newton convergence for this Robin problem)

Main result 2/3

Theorem. (**H.**, Optim. Lett. 2021)

- Suff. many measurements are taken if $\lambda_{\max}(F'(z_{j,k})d_j) > 0$ for
$$z_{j,k} := \frac{a}{2}e_j' + \left(a + k\frac{a}{4(n-1)}\right)e_j \in \mathbb{R}^n_+, \quad d_j := \frac{2b-a}{a}(n-1)e_j' - \frac{1}{2}e_j \in \mathbb{R}^n,$$
with $j = 1, \ldots, n, \quad k = 1, \ldots, \lceil \frac{4(n-1)b}{a} \rceil - 4n + 5.$
- This criterion is fulfilled if $(g_j)_{j=1}^{\infty}$ has dense span in $L^2(\partial\Omega)$, and sufficiently many g_j are used.

($e_j \in \mathbb{R}^n$: j-th unit vector, $e_j' := \mathbb{1} - e_j \in \mathbb{R}^n$: negated j-th unit vector)

$\rightsquigarrow$ Explicit, easy-to-check criterion whether a desired resolution can be achieved with a certain number of measurements

Achievable resolution can be characterized.

Main result 3/3

Theorem. (**H.**, Optim. Lett. 2021)

- Let the criterion hold with lower bound $\lambda > 0$.
- Let $\delta > 0$, and $Y^\delta \in \mathbb{R}^{m\times m}$ be symmetric with $\|\hat{Y} - Y^\delta\|_2 \leq \delta$.

Then there exist solutions of

$$\texttt{minimize} \quad \|\sigma\|_1 = \sum_{j=1}^{n} \sigma_j \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \preceq Y^\delta + \delta I.$$

and every such minimum σ^δ fulfills

$$\|\hat{\sigma} - \sigma^\delta\|_\infty \leq \frac{2\delta(n-1)}{\lambda}$$

Explicit error estimates, convergence for $\delta \to 0$.

Proof ingredients & possible generalizations

- Monotonicity & Convexity: $F: \mathbb{R}^n_+ \to \mathbb{S}_m \subset \mathbb{R}^{m\times m}$ fulfills

$$F'(\sigma)d \leq 0 \qquad \text{for all } \sigma \in \mathbb{R}^n_+,\ 0 \leq d \in \mathbb{R}^n$$
$$F(\tau) - F(\sigma) \geq F'(\sigma)(\tau - \sigma) \qquad \text{for all } \sigma, \tau \in \mathbb{R}^n_+$$

$\rightsquigarrow$ holds for general elliptic PDEs (**H.**, Jahresber. DMV, 2021)

- Localized potentials: For any $C > 0$, there exist currents g s.t.

$$g^T(F'(\sigma)(e_j - Ce_j'))g = \int_{\Gamma_j} |\nabla u|^2 \,\mathrm{d}x - C\int_{\Gamma\setminus\Gamma_j} |\nabla u|^2 > 0$$

$\Longrightarrow$ $\lambda_{\text{max}}(F'(z)(e_j - Ce_j')) > 0$ for suff. many measurem.

$\rightsquigarrow$ holds for many elliptic problems, but in more complicated form

Conclusions

For elliptic coefficient inverse problems

- least-squares residuum functionals may be highly non-convex
- local minima are usually useless

Possible remedy

- utilize monotonicity & convexity with respect to Loewner order
- utilize localized potentials to control directional derivatives

For an inverse Robin coefficient problem we can obtain

- equivalent reformulation as convex semidefinite program
- globally convergent solution algorithms
- explicit characterizations of achievable resolution
- explicit error estimates for noisy data