Uniqueness and global convergence for inverse coefficient problems with finitely many measurements

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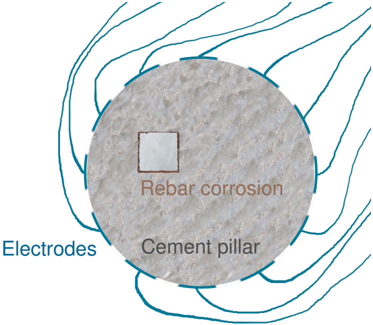
Electrical impedance tomography for corrosion detection



Non-destructive EIT-based corrosion detection:

- Apply electric currents on outer boundary $\partial\Omega$
- Measure necessary voltages

$\rightsquigarrow$ Detect corrosion on inner boundary $\Gamma = \partial D$

Idealized mathematical model: Robin PDE

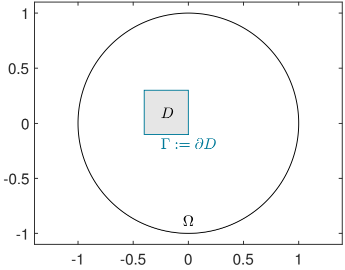
Electric potential $u: \ \Omega \to \mathbb{R}$ solves

(1) $\Delta u = 0$ in $\Omega \setminus \Gamma$,

(2) $\partial_\nu u|_{\partial\Omega} = g$ on $\partial\Omega$,

(3) $[\![u]\!]_\Gamma = 0$ on Γ,

(4) $[\![\partial_\nu u]\!]_\Gamma = \sigma u$ on Γ



Inverse Problem: Recover σ from Neumann-to-Dirichlet-Operator

$$\Lambda(\sigma): \ L^2(\partial\Omega) \to L^2(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves Robin PDE (1)–(4).

(Similar to but simpler than famous Calderón Problem)

Finitely many measurements and unknowns

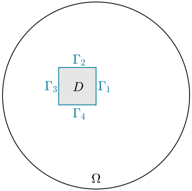
- Finitely many measurements:

$$\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \,\mathrm{d}s \quad \text{for finitely many } g_1, \ldots, g_m$$

- Finite desired resolution:

$$\sigma = \sum_{j=1}^{n} \sigma_j \chi_{\Gamma_j} \quad \text{with } \sigma_j \in \mathbb{R},\ j = 1, \ldots, n$$

with partition $\Gamma = \bigcup_{j=1}^{n} \Gamma_j$



- A-priori bounds: $\sigma := (\sigma_1, \ldots, \sigma_n)^T \in [a,b]^n$, $b > a > 0$ known

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Finite-dimensional non-linear inverse problem (IP): Determine

$$\sigma = (\sigma_j)_{j=1}^{n} \in [a,b]^n \quad \text{from} \quad F(\sigma) := \left(\int_{\partial\Omega} g_j \Lambda(\sigma) g_k \,\mathrm{d}s \right)_{j,k=1}^{m} \in \mathbb{R}^{m \times m}$$

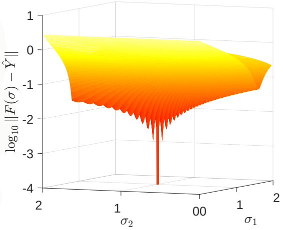
Data fitting approaches

(IP) Determine $\hat{\sigma} \in [a,b]^n \subset \mathbb{R}^n$ from $\hat{Y} := F(\hat{\sigma}) \in \mathbb{R}^{m\times m}$

- Natural approach: Least-squares data fitting

$$\texttt{minimize} \quad \|F(\sigma) - \hat{Y}\|_{\mathsf{F}}^2 \quad \textit{(+ Regularization)}$$

$\rightsquigarrow$ High-dim., non-convex minimization. Problem of local minima.



(Example from Calderón problem)

Next slides: Equivalent convex reformulation of (IP)

Main result 1/3

Theorem. (**H.**, Optim. Lett. 2021)
If sufficiently many measurements are taken, then

- $\hat{Y} := F(\hat{\sigma}) \in \mathbb{R}^{m\times m}$ uniquely determines $\hat{\sigma} \in [a,b]^n$.
- $\hat{\sigma}$ is the unique solution of

$$\texttt{minimize} \quad \|\sigma\|_1 = \sum_{j=1}^{n} \sigma_j \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \leq \hat{Y}.$$

- The constraint set $\sigma \in [a,b]^n,\ F(\sigma) \leq \hat{Y}$ is convex.

⇝ $\hat{\sigma}$ is the lower left corner of the convex constraint set

⇝ Problem can be solved by convex semidefinite programming

Global convergence is feasible.

(**H.**, Numer. Math. 2020: Global Newton convergence for this Robin problem)

Main result 2/3

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Theorem. (**H.**, Optim. Lett. 2021)

- Suff. many measurements are taken if $\lambda_{\max}(F'(z_{j,k})d_j) > 0$ for
  $$z_{j,k} := \tfrac{a}{2}e'_j + \left(a + k\tfrac{a}{4(n-1)}\right)e_j \in \mathbb{R}^n_+, \quad d_j := \tfrac{2b-a}{a}(n-1)e'_j - \tfrac{1}{2}e_j \in \mathbb{R}^n,$$
  with $j = 1, \dots, n, \quad k = 1, \dots, \lceil \frac{4(n-1)b}{a} \rceil - 4n + 5.$
- This criterion is fulfilled if $(g_j)_{j=1}^{\infty}$ has dense span in $L^2(\partial\Omega)$, and sufficiently many g_j are used.

($e_j \in \mathbb{R}^n$: j-th unit vector, $e'_j := \mathbb{1} - e_j \in \mathbb{R}^n$: negated j-th unit vector)

$\rightsquigarrow$ Explicit, easy-to-check criterion whether a desired resolution can be achieved with a certain number of measurements

Achievable resolution can be characterized.

Main result 3/3

Theorem. (**H.**, Optim. Lett. 2021)

- Let the criterion hold with lower bound $\lambda > 0$.
- Let $\delta > 0$, and $Y^\delta \in \mathbb{R}^{m\times m}$ be symmetric with $\|\hat{Y} - Y^\delta\|_2 \leq \delta$.

Then there exist solutions of

$$\texttt{minimize} \quad \|\sigma\|_1 = \sum_{j=1}^{n} \sigma_j \quad \texttt{s.t.} \quad \sigma \in [a,b]^n,\ F(\sigma) \preceq Y^\delta + \delta I.$$

and every such minimum σ^δ fulfills

$$\|\hat{\sigma} - \sigma^\delta\|_\infty \leq \frac{2\delta(n-1)}{\lambda}$$

Explicit error estimates, convergence for $\delta \to 0$.

Proof ingredients & possible generalizations

- **Monotonicity & Convexity:** $F: \mathbb{R}^n_+ \to \mathbb{S}_m \subset \mathbb{R}^{m\times m}$ fulfills

$$\begin{aligned} F'(\sigma)d &\leq 0 && \text{for all } \sigma \in \mathbb{R}^n_+,\ 0 \leq d \in \mathbb{R}^n \\ F(\tau) - F(\sigma) &\geq F'(\sigma)(\tau - \sigma) && \text{for all } \sigma, \tau \in \mathbb{R}^n_+ \end{aligned}$$

$\rightsquigarrow$ holds for general elliptic PDEs (**H.**, Jahresber. DMV, 2021)

- **Localized potentials:** For any $C > 0$, there exist currents g s.t.

$$g^T(F'(\sigma)(e_j - Ce_j'))g = \int_{\Gamma_j} |\nabla u|^2 \,\mathrm{d}x - C \int_{\Gamma \setminus \Gamma_j} |\nabla u|^2 > 0$$

$\Longrightarrow \lambda_{\text{max}}(F'(z)(e_j - Ce_j')) > 0$ for suff. many measurem.

$\rightsquigarrow$ holds for many elliptic problems, but in more complicated form

Conclusions

For elliptic coefficient inverse problems

- least-squares residuum functionals may be highly non-convex
- local minima are usually useless

Possible remedy

- utilize monotonicity & convexity with respect to Loewner order
- utilize localized potentials to control directional derivatives

For an inverse Robin coefficient problem we can obtain

- equivalent reformulation as convex semidefinite program
- globally convergent solution algorithms
- explicit characterizations of achievable resolution
- explicit error estimates for noisy data