Uniqueness and global convergence for a discrete inverse coefficient problem

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Introduction to inverse problems

Laplace's demon

Pierre Simon Laplace (1814):

"An intellect which ... would know
all forces ... and all positions of all items,

if this intellect were also vast enough to
submit these data to analysis ...

for such an intellect nothing would be
uncertain and the future just like the past
would be present before its eyes."

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Computational Science:

If we know all necessary parameters, then we can numerically predict the outcome of an experiment (by solving mathematical formulas).

Goals:

- Prediction
- Optimization
- Inversion/Identification

Requires: Solving the laws of nature *(e.g., PDEs)*

Generic simulation problem:

Given input x calculate outcome $y = F(x)$.

$x \in X$: parameters / input *(e.g., coefficients in PDE, IC & BC)*
$y \in Y$: outcome / measurements *(e.g., solution of PDE)*
$F : X \to Y$: functional relation / model *(e.g., requires solving PDE)*

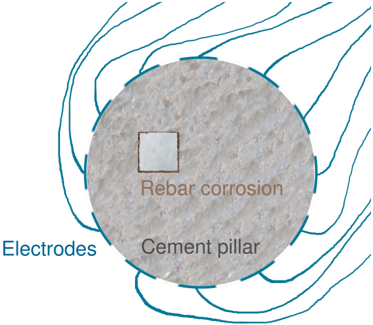
Goals:

- Prediction: Given x, calculate $y = F(x)$.
- Optimization: Find x, such that $F(x)$ is optimal.
- Inversion/Identification: Given $F(x)$, calculate x.

An inverse Robin coefficient problem

(with applications in corrosion detection)

Electrical Impedance Tomography for corrosion detection



Non-destructive EIT-based corrosion detection:

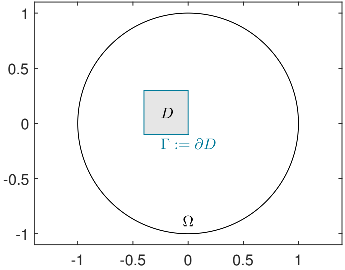
- Apply electric currents on outer boundary $\partial\Omega$
- Measure necessary voltages

$\rightsquigarrow$ Detect corrosion on inner boundary $\Gamma = \partial D$

Idealized mathematical model: Robin PDE

Electric potential $u: \ \Omega \to \mathbb{R}$ solves

$$
\begin{aligned}
\Delta u &= 0 && \text{in } \Omega \setminus \Gamma, \\
\partial_\nu u|_{\partial\Omega} &= g && \text{on } \partial\Omega, \\
[\![u]\!]_\Gamma &= 0 && \text{on } \Gamma, \\
[\![\partial_\nu u]\!]_\Gamma &= \gamma u && \text{on } \Gamma
\end{aligned}
$$



- *Applied boundary currents:* $g: \ \partial\Omega \to \mathbb{R}$
- *Corrosion coefficient:* $\gamma: \ \Gamma \to \mathbb{R}$
- *Voltage jump:* $[\![u]\!] := u^+|_\Gamma - u^-|_\Gamma$
- *Lack of electrical currents:* $[\![\partial_\nu u]\!] := \partial_\nu u^+|_\Gamma - \partial_\nu u^-|_\Gamma$
- *Measured boundary voltages:* $u|_{\partial\Omega}: \ \partial\Omega \to \mathbb{R}$

Forward and inverse problem

- Forward problem:
  Given corrosion $\gamma \in L^\infty_+(\Gamma)$ and applied currents $g \in L^2(\partial\Omega)$, predict/simulate voltage measurements $u_\gamma^{(g)}|_{\partial\Omega} \in L^2(\partial\Omega)$.
  (Existence & Uniqueness follow from standard Lax-Milgram argument)

- Inverse problem:
  Given voltages $u_\gamma^{(g)}|_{\partial\Omega} \in L^2(\partial\Omega)$ for several $g \in L^2(\partial\Omega)$, reconstruct corrosion coefficient $\gamma \in L^\infty_+(\Gamma)$.

Can we recover coefficient $\gamma \in L^\infty_+(\Gamma)$ in Robin PDE from Dirichlet and Neumann boundary values $(\partial_\nu u|_{\partial\Omega}, u|_{\partial\Omega})$?

Global uniqueness from idealized data

Theorem. (**H.**/Meftahi, SIAM J. Appl. Math. 2019)
$\gamma \in L^\infty_+(\Gamma)$ is uniquely determined by *Neumann-Dirichlet-Operator*

$$\Lambda(\gamma) : L^2(\partial\Omega) \to L^2(\partial\Omega), \quad g \mapsto u_\gamma^{(g)}|_{\partial\Omega},$$

where $u_\gamma^{(g)}$ solves Robin PDE (1)–(4).

$\rightsquigarrow$ Infinitely many measurements with infinite accuracy uniquely determine $\gamma \in L^\infty_+(\Gamma)$ with infinite resolution.

Consequences for practical applications?

Engineer vs. Mathematician

- Engineer:
  I want to determine some parameters from my measurements.
- Mathematician:
  Okay, I can solve the problem in infinite-dimensional spaces.
- Engineer:
  Why? Is my finite-dimensional problem too trivial for you? I need finite resolution from finitely many noisy measurements.
- Mathematician:
  No, your finite-dimensional problem is too hard for me.
  I can only solve the idealized infinite-dimensional version.

Towards practical applications

- Finitely many measurements:

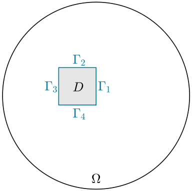
$$\int_{\partial\Omega} g_j \Lambda(\gamma) g_j \, \mathrm{d}s \quad \text{for finitely many } g_j,\ j = 1, \ldots, m$$

(power required to keep up current g_j*, electrode models yield similar expressions)*

- Finite desired resolution:

$$\gamma = \sum_{j=1}^{n} \gamma_j \chi_{\Gamma_j} \quad \text{with } \gamma_j \in \mathbb{R},\ j = 1, \ldots, n$$

with partition $\Gamma = \bigcup_{j=1}^{n} \Gamma_j$



- A-priori bounds: $\gamma := (\gamma_1, \ldots, \gamma_n)^T \in [a,b]^n$ with known $b > a > 0$.

Towards practical applications

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Finite-dimensional non-linear inverse problem: Determine

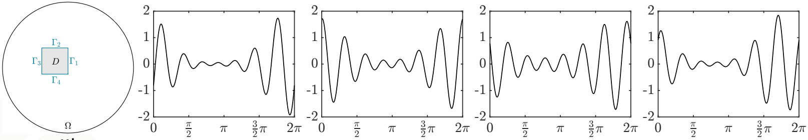
$$\gamma = (\gamma_j)_{j=1}^n \in [a,b]^n \quad \text{from} \quad F(\gamma) := \left(\int_{\partial\Omega} g_j \Lambda(\gamma) g_j \, \mathrm{d}s \right)_{j=1}^m \in \mathbb{R}^m$$

- Uniqueness: How many (and what) g_j make F injective?
- Stability/error estimates?
- How to determine γ from $F(\gamma)$? Convergence (local/global)?

Problem is much harder than the infinite-dimensional version!
But (for this simple Robin example): it can be solved.

Example result

Four unknown conductivities $\gamma_1,\dots,\gamma_4$ with a-priori bounds $1 \leq \gamma_j \leq 2$



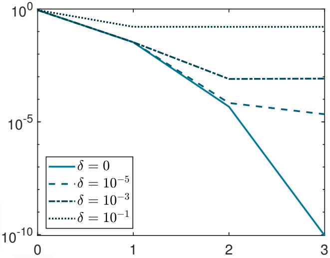
setting | g_1 | g_2 | g_3 | g_4

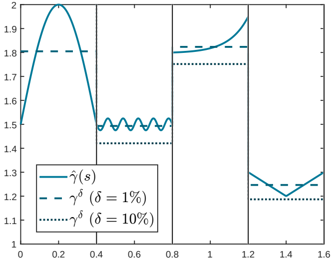
For $F(\gamma) := \left(\int_{\partial\Omega} g_j \Lambda(\gamma) g_j \,\mathrm{d}s\right)_{j=1}^4$ with these $g_1,\dots,g_4$ we can prove

(**H.**, Numer. Math. 2020)

- $F(\gamma)$ uniquely determines $\gamma \in [1,2]^4$
- $\|\gamma - \gamma'\|_\infty \leq 7.5 \dfrac{\|F(\gamma) - F(\gamma')\|_\infty}{\|F(2) - F(1)\|_\infty}$ for all $\gamma, \gamma' \in [1,2]^4$
- Newton iteration with $\gamma^{(0)} = (1,1,1,1)$ (globally!) converges.

Noisy measurements





Using $F(\gamma) := \left(\int_{\partial\Omega} g_j \Lambda(\gamma) g_j \, \mathrm{d}s\right)_{j=1}^4$ with $g_1, \ldots, g_4$ as on last slide:

- Newton convergence speed is quadratic
- For all $y^\delta \in [F(2), F(1)]^4$ there exists unique γ with $F(\gamma) = y^\delta$
  - ⇝ Lipschitz stability yields error estimate.
  - ⇝ Newton finds pcw.-const. approx. if true γ is not pcw.-const.

Rest of talk: How to construct $g_1, \ldots, g_4$ and prove such results.

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Uniqueness, stability and global Newton convergence

(for pointwise convex monotonic functions)

Pointwise convex, monotonic C^1 functions

Given $F: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $m \geq n \geq 2$, on convex open set $U \subseteq \mathbb{R}^n$.

$$F \text{ pointwise monotonic} \iff F'(x) \geq 0 \qquad \forall x \in U,$$

$$F \text{ pointwise convex} \iff F(y) - F(x) \geq F'(x)(y-x) \quad \forall x, y \in U.$$

Goal: Find criteria that ensure

- Injectivity of F
- Lipschitz continuity of F^{-1}
- Global convergence of Newton's method for $n = m$

*Results known for **inverse** monotonic convex F, i.e. $F'(x)^{-1} \geq 0$.*

*We need results for **forward** monotonic convex F, i.e. $F'(x) \geq 0$.*

Simple version of the main result

Theorem. (**H.**, Numer. Math. 2020)
$F: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $F \in C^1$, pointwise convex and monotonic. If

$$U \supset [-1,3]^n \quad \text{and} \quad F'(-e_j + 3e_j')(e_j - 3e_j') \nleq 0 \;\forall j = 1, \ldots, n,$$

then F is injective on $[0,1]^n$.

$e_j := (0 \ldots 0\ 1\ 0 \ldots 0)^T \in \mathbb{R}^n$ *unit vector,* $e_j' := 1 - e_j = (1 \ldots 1\ 0\ 1 \ldots 1)^T \in \mathbb{R}^n$

- Easy and simple-to-check criterion for injectivity
- Also yields injectivity of $F'(x)$ & Lipschitz continuity of F^{-1} with

$$L = 2\left(\min_{j=1,\ldots,n} \max_{k=1,\ldots,m} e_k^T F'(-e_j + 3e_j')\left(e_j - 3e_j'\right)\right)^{-1}$$

Sketch of proof (1/2)

Theorem. (**H.**, Numer. Math. 2020)
$F: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $F \in C^1$, pointwise convex and monotonic. If

$$U \supset [-1,3]^n \quad \text{and} \quad F'(-e_j + 3e'_j)(e_j - 3e'_j) \not\leq 0 \ \forall j = 1, \dots, n,$$

then F is injective on $[0,1]^n$.

Proof (1/2). Auxiliary result: For all $x \in [0,1]^n$,

$$e_j - 3e'_j \leq x - (-e_j + 3e'_j) \leq 2e_j - 2e'_j$$

and thus

$$\begin{aligned} 2F'(x)(e_j - e'_j) &\geq F'(x)\left(x - (-e_j + 3e'_j)\right) \geq F(x) - F(-e_j + 3e'_j) \\ &\geq F'(-e_j + 3e'_j)\left(x - (-e_j + 3e'_j)\right) \\ &\geq F'(-e_j + 3e'_j)(e_j - 3e'_j) \not\leq 0 \implies \underline{F'(x)(e_j - e'_j) \not\leq 0.} \end{aligned}$$

Sketch of proof (2/2)

Theorem. (**H.**, Numer. Math. 2020)
$F: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $F \in C^1$, pointwise convex and monotonic. If

$$U \supset [-1,3]^n \quad \text{and} \quad F'(-e_j + 3e_j')(e_j - 3e_j') \not\leq 0 \ \forall j = 1, \dots, n,$$

then F is injective on $[0,1]^n$.

Proof (2/2). Auxiliary result: $\forall x \in [0,1]^n$: $\underline{F'(x)(e_j - e_j') \not\leq 0}$.

Proof of injectivity: Let $x, y \in [0,1]^n$, $x \neq y$. Then $\exists j \in \{1, \dots, n\}$:

$$\frac{y-x}{\|y-x\|_\infty} \geq e_j - e_j' \quad \text{or} \quad \frac{x-y}{\|y-x\|_\infty} \geq e_j - e_j'$$

In the first case

$$F(y) - F(x) \geq F'(x)(y-x) \geq \|y-x\|_\infty F'(x)(e_j - e_j') \not\leq 0.$$

$\rightsquigarrow F(y) \neq F(x)$. Second case analogously. □

Global Newton convergence

Theorem. (**H.**, Numer. Math. 2020)
$F: U \subseteq \mathbb{R}^n \to \mathbb{R}^n$, $F \in C^1$, pointwise convex and monotonic. If

$$[-2, n(n+3)]^n \subset U \quad \text{and} \quad F'(z^{(j)})d^{(j)} \not\leq 0 \quad \text{for all } j \in \{1, \ldots, n\},$$

with $z^{(j)} := -2e_j + n(n+3)e'_j$, and $d^{(j)} := e_j - (n^2 + 3n + 1)e'_j$, then

- F is injective on $[-1, n]^n$, $F'(x)$ is invertible for all $x \in [-1, n]^n$.
- If, additionally, $F(0) \leq 0 \leq F(1)$, then there exists a unique

  $$\hat{x} \in \left(-\tfrac{1}{n-1}, 1 + \tfrac{1}{n-1}\right)^n \quad \text{with} \quad F(\hat{x}) = 0,$$

  The Newton iteration started with $x^{(0)} := 1$ converges against $\hat{x}$.

Proof and Comments

Proof.

- F injective and $F'(x)$ invertible: similar to sample result.
- Global Newton convergence:
  $F'(z^{(j)})d^{(j)} \not\leq 0 \Longrightarrow F$ is affine transf. of *inverse monotonic* (*Collatz monotone*) convex function, for which global Newton convergence is classic result.

Comments/Extensions

- Result allows to calculate Lipschitz constant of F^{-1}.
- Result can be formulated with arbitrarily small neighborhoods $U \supset [0,1]^n$ with criteria

$$F'(z^{(j,k)})d^{(j)} \not\leq 0 \quad \forall j \in \{1,\ldots,n\},\ k \in \{1,\ldots,K\}.$$

Back to the Robin interface problem

- Monotonicity relations (Kang/Seo/Sheen 97, Ikehata 98, **H.**/Ullrich 13)

$$F(\gamma) := \left(\int_{\partial\Omega} g_j \Lambda(\gamma) g_j \, \mathrm{d}s \right)_{j=1}^{m} \in \mathbb{R}^m$$

  is pointw. convex and monot. decreasing for any choice of g_j.

- $F \in C^1$, directional derivatives fulfill, e.g.

$$F'(\gamma)(-e_j + 3e_j') = \left(\int_{\Gamma_j} |u_\gamma^{g_k}|^2 \, \mathrm{d}s - 3 \int_{\Gamma \setminus \Gamma_j} |u_\gamma^{g_k}|^2 \, \mathrm{d}s \right)_{k=1}^{n} \in \mathbb{R}^n$$

$\rightsquigarrow$ *$F'(z^{(j)})d^{(j)} \nleq 0$ if $u_{z^{(j)}}^{g_j}$ has high energy on Γ_j and low on $\Gamma \setminus \Gamma_j$.*

Back to the Robin interface problem

$\rightsquigarrow F'(z^{(j)})d^{(j)} \not\leq 0$ *if* $u^{g_j}_{z^{(j)}}$ *has high energy on* Γ_j *and low on* $\Gamma \setminus \Gamma_j$.

- Localized potentials (**H.** 08): g_j can be chosen so that

$$F'(z^{(j)})d^{(j)} \not\leq 0 \quad \forall j$$

- Simultaneously localized potentials: g_j can be chosen so that

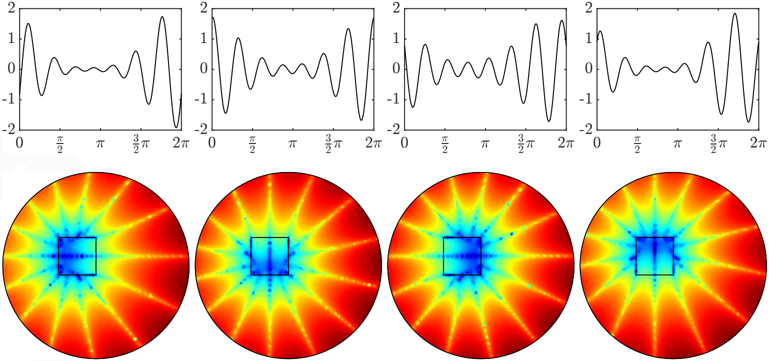
$$F'(z^{(j,k)})d^{(j)} \not\leq 0 \quad \forall j,k$$

(**H.**/Lin 20, important for treating $\gamma \in [a,b]^n$ with arbitrary $b > a > 0$)

- "g_j can be chosen": every large enough fin.-dim. subspace of $L^2(\partial\Omega)$ contains such g_j & explicit method to calculate them.

Back to the Robin interface problem

For the example result with four unknown conductivities $\gamma \in [1,2]^4$



$\log|\nabla u_\gamma^{g_1}|^2$ $\log|\nabla u_\gamma^{g_2}|^2$ $\log|\nabla u_\gamma^{g_3}|^2$ $\log|\nabla u_\gamma^{g_4}|^2$

- $u_\gamma^{g_j}$ has localized energy on Γ_j for certain γ

  (More precisely: for $K = 173$ choices of γ)

Conclusions and Outlook (1/2)

For fin.-dim. inverse problems with convex monotonic functions

- simple criterion ensures uniqueness and Lipschitz stability
- also yields global Newton convergence
- criterion requires to check finitely many directional derivatives

For a discretized inverse Robin coefficient problem

- assumptions connected to monotonicity & localized potentials
- boundary currents can be found that uniquely and stably determine conductivity with global Newton convergence

Limitations/Extensions?

- Robin problem particularly simple, extension to EIT non-trivial
- Criterion not sharp, constructed currents and stability constant not optimal, high oscillations for larger number of unknowns

Conclusions and Outlook (2/2)

- Provocative claim:

  Finite-dimensional inverse coefficients problem are much harder than infinite-dimensional ones.

- Relation to classical Collatz theory:

  Elliptic PDE forward problems lead to inverse monotonic convex functions. Inverse elliptic coefficient problems lead to forward monotonic convex functions.