

Uniqueness and global convergence for a discretized inverse coefficient problem

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Introduction to inverse problems

Pierre Simon Laplace (1814):

*"An intellect which ... would know
all forces ... and all positions of all items,
if this intellect were also vast enough to
submit these data to analysis ...*

*for such an intellect nothing would be
uncertain and the future just like the past
would be present before its eyes."*



Computational Science

Computational Science:

If we know all necessary parameters, then we can numerically predict the outcome of an experiment (by solving math. formulas).

Goals:

- ▶ Prediction
- ▶ Optimization
- ▶ Inversion/Identification

Requires: Solving the laws of nature (*e.g., PDEs*)

Generic simulation problem:

Given input x , calculate outcome $y = F(x)$.

$x \in X$: parameters / input (e.g., coefficients in PDE, IC & BC)

$y \in Y$: outcome / measurements (e.g., solution of PDE)

$F : X \rightarrow Y$: functional relation / model (e.g., requires solving PDE)

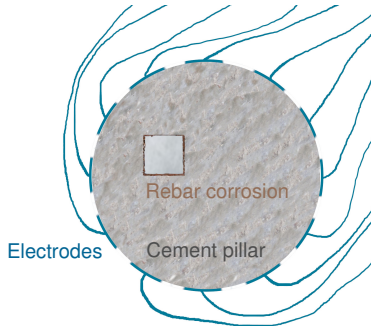
Goals:

- ▶ **Prediction:** Given x , calculate $y = F(x)$.
- ▶ **Optimization:** Find x , such that $F(x)$ is optimal.
- ▶ **Inversion/Identification:** Given $F(x)$, calculate x .

An inverse Robin coefficient problem

(with applications in corrosion detection)

Electrical Impedance Tomography for corrosion detection



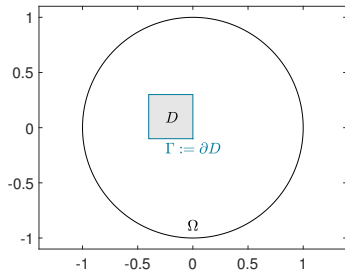
Non-destructive EIT-based corrosion detection:

- ▶ Apply electric currents on outer boundary $\partial\Omega$
- ▶ Measure necessary voltages
- ≈ Detect corrosion on inner boundary $\Gamma = \partial D$

Idealized mathematical model: Robin PDE

Electric potential $u : \Omega \rightarrow \mathbb{R}$ solves

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega \setminus \Gamma, \\ \partial_{\nu} u|_{\partial\Omega} &= g && \text{on } \partial\Omega, \\ \llbracket u \rrbracket_{\Gamma} &= 0 && \text{on } \Gamma, \\ \llbracket \partial_{\nu} u \rrbracket_{\Gamma} &= \gamma u && \text{on } \Gamma \end{aligned}$$



- ▶ *Applied boundary currents:* $g : \partial\Omega \rightarrow \mathbb{R}$
- ▶ *Corrosion coefficient:* $\gamma : \Gamma \rightarrow \mathbb{R}$
- ▶ *Voltage jump:* $\llbracket u \rrbracket := u^+|_{\Gamma} - u^-|_{\Gamma}$
- ▶ *Lack of electrical currents:* $\llbracket \partial_{\nu} u \rrbracket := \partial_{\nu} u^+|_{\Gamma} - \partial_{\nu} u^-|_{\Gamma}$
- ▶ *Measured boundary voltages:* $u|_{\partial\Omega} : \partial\Omega \rightarrow \mathbb{R}$

Forward and inverse problem

► Forward problem:

Given corrosion $\gamma \in L_+^\infty(\Gamma)$, and applied currents $g \in L^2(\partial\Omega)$,
predict/simulate voltage measurements $u_\gamma^{(g)}|_{\partial\Omega} \in L^2(\partial\Omega)$.

(Existence & Uniqueness follow from standard Lax-Milgram argument.)

► Inverse problem:

Given voltages $u_\gamma^{(g)}|_{\partial\Omega} \in L^2(\partial\Omega)$ for several $g \in L^2(\partial\Omega)$,
reconstruct corrosion coefficient $\gamma \in L_+^\infty(\Gamma)$.

*Can we recover coefficient $\gamma \in L_+^\infty(\Gamma)$ in Robin PDE
from Dirichlet and Neumann boundary values $(\partial_\nu u|_{\partial\Omega}, u|_{\partial\Omega})$?*

Global uniqueness from idealized data

Theorem. (H./Meftahi, SIAM J. Appl. Math. 2019)

$\gamma \in L_+^\infty(\Gamma)$ is uniquely determined by *Neumann-Dirichlet-Operator*

$$\Lambda(\gamma) : L^2(\partial\Omega) \rightarrow L^2(\partial\Omega), \quad g \mapsto u_\gamma^{(g)}|_{\partial\Omega},$$

where $u_\gamma^{(g)}$ solves Robin PDE (1)–(4).

- ↪ Infinitely many measurements with infinite accuracy uniquely determine $\gamma \in L_+^\infty(\Gamma)$ with infinite resolution.

Consequences for practical applications?

Engineer vs. Mathematician

- ▶ **Engineer:**
I want to determine some parameters from my measurements.
- ▶ **Mathematician:**
Okay, I can solve the problem in infinite-dimensional spaces.
- ▶ **Engineer:**
Why? Is my finite-dimensional problem too trivial for you? I need finite resolution from finitely many noisy measurements.
- ▶ **Mathematician:**
*No, your finite-dimensional problem is too hard for me.
I can only solve the idealized infinite-dimensional version.*

Towards practical applications

- Finitely many measurements:

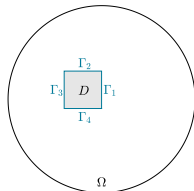
$$\int_{\partial\Omega} g_j \Lambda(\gamma) g_j \, ds \quad \text{for finitely many } g_j, j = 1, \dots, m$$

(power required to keep up current g_j , electrode models yield similar expressions)

- Finite desired resolution:

$$\gamma = \sum_{j=1}^n \gamma_j \chi_{\Gamma_j} \quad \text{with } \gamma_j \in \mathbb{R}, j = 1, \dots, n$$

with partition $\Gamma = \bigcup_{j=1}^n \Gamma_j$



- A-priori bounds: $\gamma := (\gamma_1, \dots, \gamma_n)^T \in [a, b]^n$ with known $b > a > 0$.

Towards practical applications

Finite-dimensional non-linear inverse problem: Determine

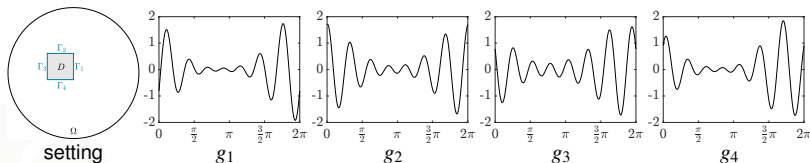
$$\gamma = (\gamma_j)_{j=1}^n \in [a, b]^n \quad \text{from} \quad F(\gamma) := \left(\int_{\partial\Omega} g_j \Lambda(\gamma) g_j \, ds \right)_{j=1}^m \in \mathbb{R}^m$$

- ▶ Uniqueness: How many (and what) g_j make F injective?
- ▶ Stability/error estimates?
- ▶ How to determine γ from $F(\gamma)$? Convergence (local/global)?

Problem is much harder than the infinite-dimensional version!
But (for this simple Robin example): it can be solved.

Example result

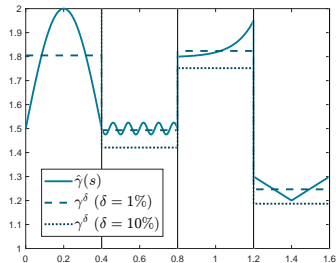
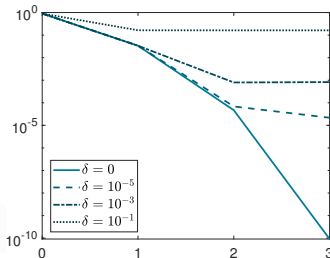
Four unknown conductivities $\gamma_1, \dots, \gamma_4$ with a-priori bounds $1 \leq \gamma_j \leq 2$



For $F(\gamma) := \left(\int_{\partial\Omega} g_j \Lambda(\gamma) g_j \, ds \right)_{j=1}^4$ with these g_1, \dots, g_4 we can prove
(H., Numer. Math. 2020)

- ▶ $F(\gamma)$ uniquely determines $\gamma \in [1, 2]^4$
- ▶ $\|\gamma - \gamma'\|_\infty \leq 7.5 \frac{\|F(\gamma) - F(\gamma')\|_\infty}{\|F(2) - F(1)\|_\infty}$ for all $\gamma, \gamma' \in [1, 2]^4$
- ▶ Newton iteration with $\gamma^{(0)} = (1, 1, 1, 1)$ (globally!) converges.

Noisy measurements



Using $F(\gamma) := \left(\int_{\partial\Omega} g_j \Lambda(\gamma) g_j \, ds \right)_{j=1}^4$ with g_1, \dots, g_4 as on last slide:

- ▶ Newton convergence speed is quadratic
- ▶ For all $y^\delta \in [F(2), F(1)]^4$ there exists unique γ with $F(\gamma) = y^\delta$
 - ↪ Lipschitz stability yields error estimate.
 - ↪ Newton finds pcw.-const. approx. if true γ is not pcw.-const.

Rest of talk: How to construct g_1, \dots, g_4 and prove such results.

Uniqueness, stability and global Newton convergence

(for pointwise convex monotonic functions)

Pointwise convex, monotonic C^1 functions

Given $F : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$, $m \geq n \geq 2$, on convex open set $U \subseteq \mathbb{R}^n$.

F pointwise monotonic $\iff F'(x) \geq 0 \quad \forall x \in U$,

F pointwise convex $\iff F(y) - F(x) \geq F'(x)(y - x) \quad \forall x, y \in U$.

Goal: Find criteria that ensure

- ▶ Injectivity of F
- ▶ Lipschitz continuity of F^{-1}
- ▶ Global convergence of Newton's method for $n = m$

*Results known for **inverse** monotonic convex F , i.e. $F'(x)^{-1} \geq 0$.*

*We need results for **forward** monotonic convex F , i.e. $F'(x) \geq 0$.*

Simple version of the main result

Theorem. (H., Numer. Math. 2020)

$F : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$, $F \in C^1$, pointwise convex and monotonic. If

$$U \supset [-1, 3]^n \quad \text{and} \quad F'(-e_j + 3e'_j)(e_j - 3e'_j) \not\leq 0 \quad \forall j = 1, \dots, n,$$

then F is injective on $[0, 1]^n$.

$e_j := (0 \dots 0 \ 1 \ 0 \dots 0)^T \in \mathbb{R}^n$ unit vector, $e'_j := 1 - e_j = (1 \dots 1 \ 0 \ 1 \dots 1)^T \in \mathbb{R}^n$

- ▶ Easy and simple-to-check criterion for injectivity
- ▶ Also yields injectivity of $F'(x)$ & Lipschitz continuity of F^{-1} with

$$L = 2 \left(\min_{j=1, \dots, n} \max_{k=1, \dots, m} e_k^T F'(-e_j + 3e'_j)(e_j - 3e'_j) \right)^{-1}$$

Sketch of proof (1/2)

Theorem. (H., Numer. Math. 2020)

$F : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$, $F \in C^1$, pointwise convex and monotonic. If

$$U \supset [-1, 3]^n \quad \text{and} \quad F'(-e_j + 3e'_j)(e_j - 3e'_j) \not\leq 0 \quad \forall j = 1, \dots, n,$$

then F is injective on $[0, 1]^n$.

Proof (1/2). Auxiliary result: For all $x \in [0, 1]^n$,

$$e_j - 3e'_j \leq x - (-e_j + 3e'_j) \leq 2e_j - 2e'_j$$

and thus

$$\begin{aligned} 2F'(x)(e_j - e'_j) &\geq F'(x)(x - (-e_j + 3e'_j)) \geq F(x) - F(-e_j + 3e'_j) \\ &\geq F'(-e_j + 3e'_j)(x - (-e_j + 3e'_j)) \\ &\geq F'(-e_j + 3e'_j)(e_j - 3e'_j) \not\leq 0 \implies \underline{F'(x)(e_j - e'_j) \not\leq 0}. \end{aligned}$$

Sketch of proof (2/2)

Theorem. (H., Numer. Math. 2020)

$F : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$, $F \in C^1$, pointwise convex and monotonic. If

$$U \supset [-1, 3]^n \quad \text{and} \quad F'(-e_j + 3e'_j)(e_j - 3e'_j) \not\leq 0 \quad \forall j = 1, \dots, n,$$

then F is injective on $[0, 1]^n$.

Proof (2/2). Auxiliary result: $\forall x \in [0, 1]^n: \overline{F'(x)(e_j - e'_j)} \not\leq 0$.

Proof of injectivity: Let $x, y \in [0, 1]^n$, $x \neq y$. Then $\exists j \in \{1, \dots, n\}$:

$$\frac{y-x}{\|y-x\|_\infty} \geq e_j - e'_j \quad \text{or} \quad \frac{x-y}{\|y-x\|_\infty} \geq e_j - e'_j$$

In the first case

$$F(y) - F(x) \geq F'(x)(y-x) \geq \|y-x\|_\infty F'(x)(e_j - e'_j) \not\leq 0.$$

$\leadsto F(y) \neq F(x)$. Second case analogously.

□

Global Newton convergence

Theorem. (H., Numer. Math. 2020)

$F : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$, $F \in C^1$, pointwise convex and monotonic. If

$$[-2, n(n+3)]^n \subset U \quad \text{and} \quad F'(z^{(j)})d^{(j)} \not\leq 0 \quad \text{for all } j \in \{1, \dots, n\},$$

with $z^{(j)} := -2e_j + n(n+3)e'_j$, and $d^{(j)} := e_j - (n^2 + 3n + 1)e'_j$, then

- ▶ F is injective on $[-1, n]^n$, $F'(x)$ is invertible for all $x \in [-1, n]^n$.
- ▶ If, additionally, $F(0) \leq 0 \leq F(1)$, then there exists a unique

$$\hat{x} \in \left(-\frac{1}{n-1}, 1 + \frac{1}{n-1}\right)^n \quad \text{with} \quad F(\hat{x}) = 0,$$

The Newton iteration started with $x^{(0)} := 1$ converges against \hat{x} .

Proof and Comments

Proof.

- ▶ F injective and $F'(x)$ invertible: similar to sample result.
- ▶ Global Newton convergence:
 $F'(z^{(j)})d^{(j)} \not\leq 0 \implies F$ is affine transf. of *inverse monotonic* (*Collatz monotone*) convex function, for which global Newton convergence is classic result.

Comments/Extensions

- ▶ Result allows to calculate Lipschitz constant of F^{-1} .
- ▶ Result can be formulated with arbitrarily small neighborhoods $U \supset [0, 1]^n$ with criteria

$$F'(z^{(j,k)})d^{(j)} \not\leq 0 \quad \forall j \in \{1, \dots, n\}, k \in \{1, \dots, K\}.$$

Back to the Robin interface problem

- Monotonicity relations (Kang/Seo/Sheen 97, Ikehata 98, H./Ullrich 13)

$$F(\gamma) := \left(\int_{\partial\Omega} g_j \Lambda(\gamma) g_j \, ds \right)_{j=1}^m \in \mathbb{R}^m$$

is pointw. convex and monot. decreasing for any choice of g_j .

- $F \in C^1$, directional derivatives fulfill, e.g.

$$F'(\gamma)(-e_j + 3e'_j) = \left(\int_{\Gamma_j} |u_\gamma^{g_k}|^2 \, ds - 3 \int_{\Gamma \setminus \Gamma_j} |u_\gamma^{g_k}|^2 \, ds \right)_{k=1}^n \in \mathbb{R}^n$$

$\leadsto F'(z^{(j)})d^{(j)} \not\leq 0$ if $u_{z^{(j)}}^{g_j}$ has high energy on Γ_j and low on $\Gamma \setminus \Gamma_j$.

Back to the Robin interface problem

$\leadsto F'(z^{(j)})d^{(j)} \not\equiv 0$ if $u_{z^{(j)}}^{g_j}$ has high energy on Γ_j and low on $\Gamma \setminus \Gamma_j$.

- ▶ Localized potentials (H. 08): g_j can be chosen so that

$$F'(z^{(j)})d^{(j)} \not\equiv 0 \quad \forall j$$

- ▶ Simultaneously localized potentials: g_j can be chosen so that

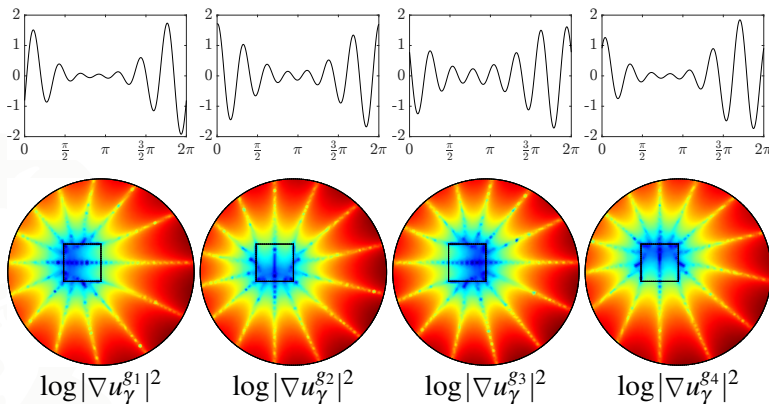
$$F'(z^{(j,k)})d^{(j)} \not\equiv 0 \quad \forall j, k$$

(H./Lin 20, important for treating $\gamma \in [a, b]^n$ with arbitrary $b > a > 0$)

- ▶ " g_j can be chosen": every large enough fin.-dim. subspace of $L^2(\partial\Omega)$ contains such g_j & explicit method to calculate them.

Back to the Robin interface problem

For the example result with four unknown conductivities $\gamma \in [1, 2]^4$



- $u_\gamma^{g_j}$ has localized energy on Γ_j for certain γ
(More precisely: for $K = 173$ choices of γ)

Conclusions and Outlook (1/2)

For fin.-dim. inverse problems with convex monotonic functions

- ▶ simple criterion ensures uniqueness and Lipschitz stability
- ▶ also yields global Newton convergence
- ▶ criterion requires to check finitely many directional derivatives

For a discretized inverse Robin coefficient problem

- ▶ assumptions connected to monotonicity & localized potentials
- ▶ boundary currents can be found that uniquely and stably determine conductivity with global Newton convergence

Limitations/Extensions?

- ▶ Robin problem particularly simple, extension to EIT non-trivial
- ▶ Criterion not sharp, constructed currents and stability constant not optimal, high oscillations for larger number of unknowns

Conclusions and Outlook (2/2)

- ▶ Provocative claim:

Finite-dimensional inverse coefficients problem are much harder than infinite-dimensional ones.

- ▶ Relation to classical Collatz theory:

Elliptic PDE forward problems lead to inverse monotonic convex functions. Inverse elliptic coefficient problems lead to forward monotonic convex functions.