Monotonicity methods for inverse coefficient problems

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Inverse Problems for Partial Differential Equations
In honor of Professor Masaru Ikehata
on the occasion of his 60th Birthday
Tokyo University of Science, Japan, August 26–28, 2018

Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

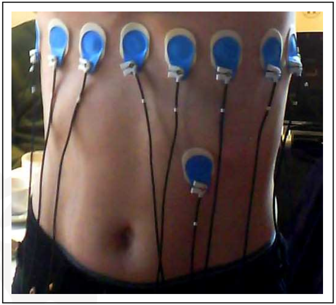
$$\{(u|_{\partial\Omega}, \sigma\partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves } (1)\}\,?$$

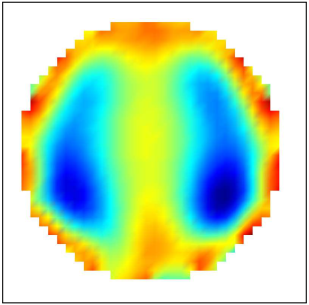
Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma): \; L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma\partial_\nu u|_{\partial\Omega} = g$.

Application: Electrical impedance tomography





- Apply electric currents on subject's boundary
- Measure necessary voltages

$\rightsquigarrow$ Reconstruct conductivity inside subject

Inversion of $\sigma \mapsto \Lambda(\sigma)$?

Generic solvers for non-linear inverse problems:

- Linearize and regularize:

$$\Lambda_{\text{meas}} \approx \Lambda(\sigma) \approx \Lambda(\sigma_0) + \Lambda'(\sigma_0)(\sigma - \sigma_0).$$

  σ_0: Initial guess or reference state (e.g. exhaled state)

  $\rightsquigarrow$ Linear inverse problem for σ

  (Solve using linear regularization method, repeat for Newton-type algorithm.)

- Regularize and linearize:

  E.g., minimize non-linear Tikhonov functional

$$\|\Lambda_{\text{meas}} - \Lambda(\sigma)\|^2 + \alpha \|\sigma - \sigma_0\|^2 \to \text{min!}$$

Advantages of generic optimization-based solvers:

- Very flexible, additional data/unknowns easily incorporated
- Problem-specific regularization can be applied

  (e.g., total variation penalization, stochastic priors, etc.)

Inversion of $\sigma \mapsto \Lambda(\sigma)$?

Problems with generic optimization-based solvers

- High computational cost
  - Evaluations of $\Lambda(\cdot)$ and $\Lambda'(\cdot)$ require PDE solutions.
  - PDE solutions too expensive for real-time imaging

- Convergence unclear
  (Validity of TCC/Scherzer-condition is a long-standing open problem for EIT.)
  - Convergence against true solution for exact meas. Λ_{meas}?
    (in the limit of infinite computation time)
  - Convergence against true solution for noisy meas. $\Lambda_{\text{meas}}^{\delta}$?
    (in the limit of $\delta \to 0$ and infinite computation time)
  - Global convergence? Resolution estimates for realistic noise?

Is there any specific problem structure that we can use to derive convergent algorithms?

Ikehata-Kang-Seo-Sheen Monotonicity

For two conductivities $\sigma_0, \sigma_1 \in L^\infty(\Omega)$:

$$\sigma_0 \leq \sigma_1 \quad \Longrightarrow \quad \Lambda(\sigma_0) \geq \Lambda(\sigma_1)$$

This follows from *(Kang/Seo/Sheen 1997, Ikehata 1998)*

$$\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \geq \int_{\partial\Omega} g\,(\Lambda(\sigma_0) - \Lambda(\sigma_1))\,g \geq \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)|\nabla u_0|^2$$

for all solutions u_0 of

$$\nabla \cdot (\sigma_0 \nabla u_0) = 0, \quad \sigma_0 \partial_\nu u_0|_{\partial\Omega} = g.$$

The monotonicity method for inclusion detection in EIT

Monotonicity method

Sample inclusion detection problem (for ease of presentation)

- $\sigma_0 = 1$
- $\sigma = 1 + \chi_D$
- D open, $\overline{D} \subseteq \Omega$, $\Omega \setminus \overline{D}$ connected

All of the following also holds for

- σ_0 pcw. analytic and known,
- $\sigma = \sigma_0 + \kappa\chi_D$ with $\kappa \in L^\infty_+(D)$,
- in any dimension $n \geq 2$,
- for partial boundary data on open subset $\Gamma \subseteq \partial\Omega$.

Monotonicity method

Sample inclusion detection problem

- $\sigma_0 = 1$, $\sigma = 1 + \chi_D$, D open, $\overline{D} \subseteq \Omega$, $\Omega \setminus \overline{D}$ connected

Monotonicity

- $\tau \leq \sigma \quad \Longrightarrow \quad \Lambda(\tau) \geq \Lambda(\sigma)$

Monotonicity-based inclusion detection (Tamburrino/Rubinacci 2002):

$$B \subseteq D \quad \Longrightarrow \quad 1 + \chi_B \leq \sigma \quad \Longrightarrow \quad \Lambda(1 + \chi_B) \geq \Lambda(\sigma)$$

Algorithm:

- Mark all balls B with $\Lambda(1 + \chi_B) \geq \Lambda(\sigma)$
- Result: upper bound of D.

Only an upper bound? Converse monotonicity relation?

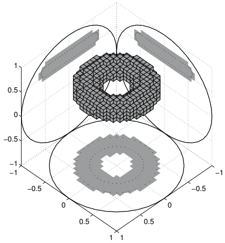
Monotonicity method (for simple test example)

Theorem *(**H.**/Ullrich, 2013)*

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1+\chi_B) \geq \Lambda(\sigma).$$

For faster implementation:

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1) + \tfrac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$



Shape can be reconstructed by linearized monotonicity tests.

Next slides: Proof using monotonicity & localized potentials

Localized potentials

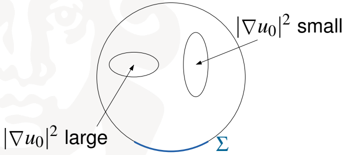
Theorem *(**H.**, 2008)*

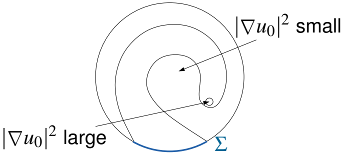
Let σ_0 fulfill unique continuation principle (UCP),

$$\overline{D_1} \cap \overline{D_2} = \emptyset, \quad \text{and} \quad \Omega \setminus (\overline{D}_1 \cup \overline{D}_2) \text{ be connected with } \Sigma.$$

Then there exist solutions $u_0^{(k)}$, $k \in \mathbb{N}$ with

$$\int_{D_1} \left| \nabla u_0^{(k)} \right|^2 \mathrm{d}x \to \infty \quad \text{and} \quad \int_{D_2} \left| \nabla u_0^{(k)} \right|^2 \mathrm{d}x \to 0.$$





Proof of converse monotonicity (for simple test example)

- Monotonicity: For all solutions u_B for conductivity $\sigma_B := 1 + \chi_B$

$$\int_D |\nabla u_B|^2 \, \mathrm{d}x - \int_B |\nabla u_B|^2 \, \mathrm{d}x = \int_\Omega (\sigma - \sigma_B) |\nabla u_B|^2 \, \mathrm{d}x$$
$$\geq \int_{\partial\Omega} g \left(\Lambda(1+\chi_B) - \Lambda(\sigma) \right) g$$
$$\geq \int_\Omega \frac{\sigma_B}{\sigma} (\sigma - \sigma_B) |\nabla u_B|^2 \, \mathrm{d}x \geq \int_D \frac{1}{2} |\nabla u_B|^2 \, \mathrm{d}x - \int_B |\nabla u_B|^2 \, \mathrm{d}x$$

- Localized potentials: If $B \not\subseteq D$ then we find $u_B^{(k)}$ with

$$\int_D |\nabla u_B^{(k)}|^2 \, \mathrm{d}x \to 0, \quad \int_B |\nabla u_B^{(k)}|^2 \, \mathrm{d}x \to \infty.$$

$\leadsto$ $B \not\subseteq D$ implies $\Lambda(1+\chi_B) - \Lambda(\sigma) \not\geq 0$. $\square$

Monotonicity method

H./Ullrich, SIAM J. Math. Anal. 2013:

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1+\chi_B) \geq \Lambda(\sigma) \quad \Longleftrightarrow \quad \Lambda(1) + \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma)$$

- Yields theoretical uniqueness result
- Simple to implement, no PDE solutions
- Similar comput. cost as single Newton (linearization) step
- Rigorously detects unknown shape for exact data
- Convergence for noisy data $\Lambda^\delta_{\text{meas}} \to \Lambda(\sigma) - \Lambda(1)$:

$$R(\Lambda^\delta_{\text{meas}}, \delta, B) := \begin{cases} 1 & \text{if } \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda^\delta_{\text{meas}} - \delta I \\ 0 & \text{else.} \end{cases}$$

Then $R(\Lambda^\delta_{\text{meas}}, \delta, B) \to 1$ iff $B \subseteq D$.

Monotonicity-based regularization of optimization-based methods

Monotonicity method

Quantitative, pixel-based variant of monotonicity method:

- Pixel partition $\Omega = \bigcup_{k=1}^{m} P_k$
- Quantitative monotonicity tests

$$\beta_k \in [0,\infty] \text{ max. values s.t. } \beta_k \Lambda'(1)\chi_{P_k} \geq \Lambda(\sigma) - \Lambda(1)$$

$$\beta_k^\delta \in [0,\infty] \text{ max. values s.t. } \beta_k^\delta \Lambda'(1)\chi_{P_k} \geq \Lambda_{\text{meas}}^\delta - \delta I$$

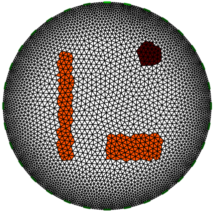
"Raise conductivity in each pixel until monotonicity test fails."

- By theory of monotonicity method:

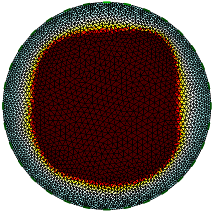
$$\beta_k^\delta \to \beta_k \quad \text{and} \quad \beta_k \text{ fulfills } \begin{cases} \beta_k = 0 & \text{if } P_k \not\subseteq D \\ \beta_k \geq \frac{1}{2} & \text{if } P_k \subseteq D \end{cases}$$

Plotting β_k^δ shows true inclusions up to pixel partition.

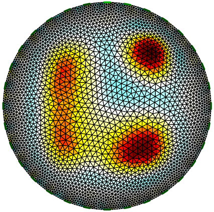
Realistic example (32 electrodes, 1% noise)



True image



Monotonicity method



Standard linearized method

- Monotonicity method rigorously converges for $\delta \to 0 \ldots$
- ...but the heuristic standard linearized method works much better for realistic scenarios.

Can we improve the monotonicity method without loosing convergence?

Monotonicity-based regularization

- Standard linearized methods for EIT: Minimize

$$\|\Lambda'(1)\kappa - (\Lambda(\sigma) - \Lambda(1))\|^2 + \alpha \|\kappa\|^2 \to \text{min!}$$

Choice of norms heuristic. No convergence theory!

- Monotonicity-based regularization: Minimize

$$\|\Lambda'(1)\kappa - (\Lambda(\sigma) - \Lambda(1))\|_{\mathrm{F}} \to \text{min!}$$

under the constraint $\kappa|_{P_k} = \text{const.}$, $0 \leq \kappa|_{P_k} \leq \min\{\frac{1}{2}, \beta_k\}$.

($\|\cdot\|_F$: Frobenius norm of Galerkin projektion to finite-dimensional space)

Theorem (**H.**/Mach, Inverse Problems 2016)

- There exists unique minimizer $\hat{\kappa}$ and

$$P_k \subseteq \operatorname{supp} \hat{\kappa} \quad \Longleftrightarrow \quad P_k \subseteq \operatorname{supp}(\sigma - 1).$$

- Minimizer fulfills $\hat{\kappa} = \sum_{k=1}^m \min\{1/2, \beta_k\} \chi_{P_k}$

Monotonicity-based regularization

For noisy measurements $\Lambda^\delta_{\text{meas}} \approx \Lambda(\sigma) - \Lambda(1)$:

- Use regularized monotonicity tests

$$\beta_k^\delta \in [0,\infty] \text{ max. values s.t. } \beta_k^\delta \Lambda'(1)\chi_{P_k} \geq \Lambda^\delta_{\text{meas}} - \delta I$$

($\delta > 0$: noise level in $\mathcal{L}(L^2_\diamond(\partial\Omega))$-norm)

- Minimize

$$\|\Lambda'(1)\kappa^\delta - \Lambda^\delta_{\text{meas}}\|_{\text{F}} \to \text{min!}$$

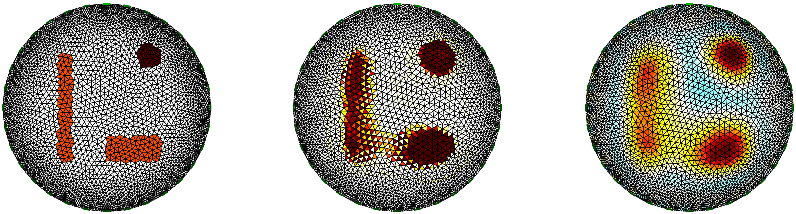
under the constraint $\kappa^\delta|_{P_k} = \text{const.}$, $0 \leq \kappa^\delta|_{P_k} \leq \min\{\frac{1}{2}, \beta_k^\delta\}$.

Theorem (**H.**/Mach, Inverse Problems 2016)

- There exist minimizers κ^δ and $\kappa^\delta \to \hat{\kappa}$ for $\delta \to 0$.

Monotonicity-regularized solutions converge against correct shape.

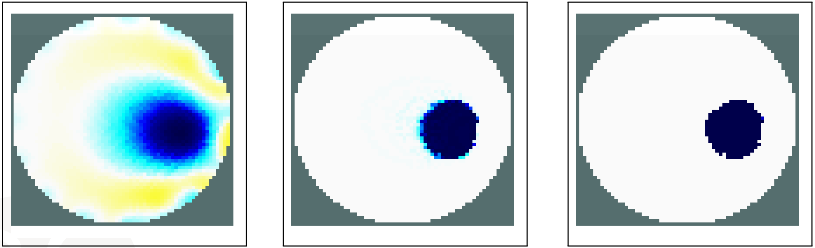
Realistic example (32 electrodes, 1% noise)



True image | Monotonicity regularized method | Standard linearized method

- Monotonicity regularized method rigorously converges and is up to par with (outperforms?) heuristic standard linearized method.

Phantom data example



standard | monoton.-regularized (Matlab quadprog) | monoton.-regularized (cvx package)

Monotonicity-regularization vs. community standard

(**H.**/Mach, Trends Math. 2018)

- EIDORS: `http://eidors3d.sourceforge.net` (Adler/Lionheart)
- EIDORS standard solver: linearized method with Tikhonov regularization
- Dataset: `iirc_data_2006` (Woo et al.): 2cm insulated inclusion in 20cm tank
  - using interpolated data on active electrodes (**H.**, Inverse Problems 2015)

Monotonicity-based Uniqueness and Lipschitz-stability

Uniqueness

Monotonicity & localized potentials yield uniqueness results:

- Non-linear Calderón problem: (Kohn/Vogelius 1985, **H.**/Seo 2010)
  If $\sigma_1 \in L^\infty_+(\Omega)$ fulfills (UCP) and $\sigma_2 - \sigma_1$ is pcw. analytic then

  $$\Lambda(\sigma_1) - \Lambda(\sigma_2) \quad \text{implies} \quad \sigma_1 = \sigma_2.$$

- Linearized Calderón problem: (**H.**/Seo 2010)
  If $\sigma_1 \in L^\infty_+(\Omega)$ fulfills (UCP) and $\kappa \in L^\infty(\Omega)$ is pcw. analytic then

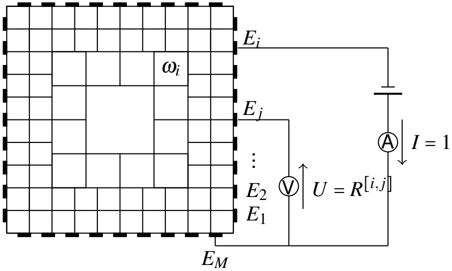
  $$\Lambda'(\sigma_1)\kappa = 0 \quad \text{implies} \quad \kappa = 0.$$

- Linearized & discretized Calderón problem: (Lechleiter/Rieder 2008)
  With enough electrodes, the linearized Calderón problem with CEM is uniquely solvable in fin.-dim. subspaces of pcw. analytic functions (e.g., pcw. polynomials of fixed degree on fixed partition).

Nonlinear Calderón problem with electrode measurements

Complete Electrode Model

$$\nabla\cdot(\sigma\nabla u)=0 \quad \text{in } \Omega$$
$$u|_{E_m}+z\sigma\partial_\nu u|_{E_m}=\text{const.}=:U_m$$
$$\int_{E_m}\sigma\partial_\nu u|_{E_m}\,\mathrm{d}s=J_m$$
$$\sigma\partial_\nu u=0 \quad \text{else}$$



Current-to-Voltage operator

$$R_M(\sigma):\ \mathbb{R}^M_\diamond\to\mathbb{R}^M_\diamond,\quad (J_1,\ldots,J_M)\mapsto(U_1,\ldots,U_M).$$

Can we uniquely and stably recover σ from $R(\sigma)$?

Uniqueness and Lipschitz-stability

Assumptions:

- Increasing number of electrodes fulfilling Hyvönen conditions
- $\mathcal{F}$: finite-dimensional subset of pcw.-analytic functions

  (e.g., pcw. constant on fixed a-priori known partition)
- Known background conductivity:

  $\exists U$ nbr.hood of $\partial\Omega$, $\sigma_0 \in C^\infty$, so that $\sigma|_U = \sigma_0|_U$ for all $\sigma \in \mathcal{F}$
- A-prior known bounds

$$\mathcal{F}_{[a,b]} := \{\sigma \in \mathcal{F} : \; a \leq \sigma(x) \leq b \text{ for all } x \in \Omega\}$$

Theorem. (**H**, submitted) $\exists N \in \mathbb{N}, \; c > 0$:

$$\|R_M(\sigma_1) - R_M(\sigma_2)\|_{\mathcal{L}(\mathbb{R}^M_\diamond)} \geq c \, \|\sigma_1 - \sigma_2\|_{L^\infty(\Omega)} \quad \forall \sigma_1, \sigma_2 \in \mathcal{F}_{[a,b]}, M \geq N.$$

Proof (main ideas)

- Monotonicity (**H**/Ullrich, 2015)

$$\langle\big(R'(\sigma_2)(\sigma_1-\sigma_2)\big)J,J\rangle_M = \int_\Omega (\sigma_2-\sigma_1)|\nabla u_{\sigma_2}^{(J)}|^2\,\mathrm{d}x$$
$$\leq \langle(R_M(\sigma_1)-R_M(\sigma_2))J,J\rangle_M .$$

$\leadsto$ Lower bound on Lipschitz stability

$$\|R_M(\sigma_1)-R_M(\sigma_2)\| \geq \|\sigma_1-\sigma_2\| \inf_{\substack{(\tau_1,\tau_2,\kappa)\\ \in\mathcal{F}_{[a,b]}\times\mathcal{F}_{[a,b]}\times\mathcal{K}}} \sup_{\substack{J\in\mathbb{R}^M_\diamond\\ \|J\|=1}} f_M(\tau_1,\tau_2,\kappa,J),$$

$$f_M(\tau_1,\tau_2,\kappa,J) := \max\left\{\left\langle\big(R_M'(\tau_1)\kappa\big)J,J\right\rangle, -\left\langle\big(R_M'(\tau_2)\kappa\big)J,J\right\rangle\right\},$$

- Relation to NtD-operators, localized potentials & compactness

$$\inf_{\substack{(\tau_1,\tau_2,\kappa)\\ \in\mathcal{F}_{[a,b]}\times\mathcal{F}_{[a,b]}\times\mathcal{K}}} \sup_{\substack{J\in\mathbb{R}^M_\diamond\\ \|J\|=1}} f_M(\tau_1,\tau_2,\kappa,J) > 0$$

$\square$

Conclusions

Ikehata-Kang-Seo-Sheen Monotonicity yields

- fundamental relation between measurements and unknowns,
- convergent inclusion detection methods,
- rigorous regularizers for residuum-based methods,
- theoretical uniqueness and Lipschitz stability results.

Approach can be extended

- to partial boundary data, independently of dimension $n \geq 2$,
- to stochastic settings,
- at least partially to closely related problems
  (diffuse optical tomography, magnetostatics, inverse scattering, eddy-current equations, p-Laplacian, fractional diffusion, ...)