Monotonicity methods for inverse coefficient problems

Bastian von Harrach

http://numerical.solutions

Institute of Mathematics, Goethe University Frankfurt, Germany

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Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \subset \mathbb{R}^d \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

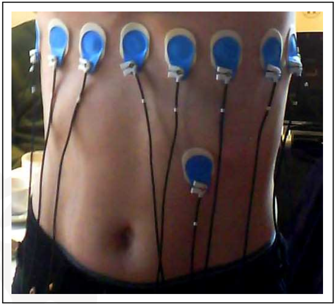
$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves } (1)\}\ ?$$

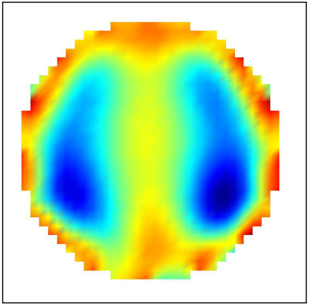
Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma) : L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Application: Electrical impedance tomography





- Apply electric currents on subject's boundary
- Measure necessary voltages

⇝ Reconstruct conductivity inside subject

Generic approaches for inverting $\sigma \mapsto \Lambda(\sigma)$

- Penalty-based regularization: Minimize Tikhonov functional

$$\|\Lambda_{\text{meas}} - \Lambda(\sigma)\|^2 + \alpha \|\sigma - \sigma_0\|^2 \to \text{min!}$$

σ_0: Initial guess or known reference state (e.g. exhaled state)

- Deep learning based methods:

Given training data $\{(\sigma_n, \Lambda(\sigma_n)) : n = 1, \ldots, N\}$ minimize

$$\sum_{n=1}^{N} \|\sigma_n - f(\Lambda(\sigma_n))\|^2 \to \text{min!}$$

over all functions $f \in \mathbb{DL}$ described by DL-network.

Advantages: Very flexible, additional data/unknowns easily added

Disadvantages: Almost no rigorous theory (convergence, resolution, ...)

Is there any specific problem structure that we can use to derive convergent algorithms?

Ikehata-Kang-Seo-Sheen Monotonicity

For two conductivities $\sigma_0, \sigma_1 \in L^\infty(\Omega)$:

$$\sigma_0 \leq \sigma_1 \quad \Longrightarrow \quad \Lambda(\sigma_0) \geq \Lambda(\sigma_1)$$

This follows from *(Kang/Seo/Sheen 1997, Ikehata 1998)*

$$\int_\Omega (\sigma_1 - \sigma_0) |\nabla u_0|^2 \geq \int_{\partial\Omega} g\,(\Lambda(\sigma_0) - \Lambda(\sigma_1))\,g \geq \int_\Omega \frac{\sigma_0}{\sigma_1} (\sigma_1 - \sigma_0) |\nabla u_0|^2$$

for all solutions u_0 of

$$\nabla \cdot (\sigma_0 \nabla u_0) = 0, \quad \sigma_0 \partial_\nu u_0|_{\partial\Omega} = g.$$

The monotonicity method for inclusion detection in EIT

Sample inclusion detection problem (for ease of presentation)

- $\sigma_0 = 1$
- $\sigma = 1 + \chi_D$
- D open, $\overline{D} \subseteq \Omega$, $\Omega \setminus \overline{D}$ connected

All of the following also holds for

- σ_0 pcw. analytic and known,
- $\sigma = \sigma_0 + \kappa\chi_D$ with $\kappa \in L^\infty_+(D)$,
- in any dimension $n \geq 2$,
- for partial boundary data on open subset $\Gamma \subseteq \partial\Omega$.

Monotonicity method

Sample inclusion detection problem

- $\sigma_0 = 1$, $\sigma = 1 + \chi_D$, D open, $\overline{D} \subseteq \Omega$, $\Omega \setminus \overline{D}$ connected

Monotonicity

- $\tau \leq \sigma \quad \Longrightarrow \quad \Lambda(\tau) \geq \Lambda(\sigma)$

Monotonicity-based inclusion detection (Tamburrino/Rubinacci 2002):

$$B \subseteq D \quad \Longrightarrow \quad 1 + \chi_B \leq \sigma \quad \Longrightarrow \quad \Lambda(1 + \chi_B) \geq \Lambda(\sigma)$$

Algorithm:

- Mark all balls B with $\Lambda(1 + \chi_B) \geq \Lambda(\sigma)$
- Result: upper bound of D.

Only an upper bound? Converse monotonicity relation?

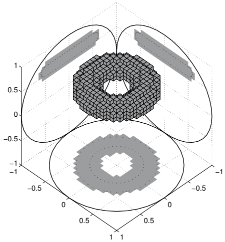
Monotonicity method (for simple test example)

Theorem *(**H.**/Ullrich, 2013)*

$$B \subseteq D \iff \Lambda(1+\chi_B) \geq \Lambda(\sigma).$$

For faster implementation:

$$B \subseteq D \iff \Lambda(1) + \tfrac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$



Shape can be reconstructed by linearized monotonicity tests.

Idea of proof: Combine monotonicity inequality:

$$\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \geq \int_{\partial\Omega} g\,(\Lambda(\sigma_0) - \Lambda(\sigma_1))\,g \geq \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)|\nabla u_0|^2$$

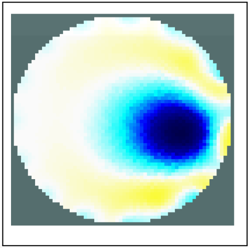
with localized potentials *(**H.**, 2008)*:

$$\int_{D_1} \left|\nabla u_0^{(k)}\right|^2 \mathrm{d}x \to \infty \quad \text{and} \quad \int_{D_2} \left|\nabla u_0^{(k)}\right|^2 \mathrm{d}x \to 0.$$

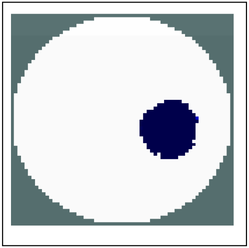
Monotonicity-based regularization

For real data: Monotonicity for regularizing residuum-based methods

- Rigorous convergence of reconstructed shape (**H.**/Mach, 2016)
- Comparison with heuristic standard for tank data (**H.**/Mach, 2018)



standard



monoton.-regularized

- EIDORS: `http://eidors3d.sourceforge.net` (Adler/Lionheart)
- EIDORS standard solver: heuristic linearized method with Tikhonov regularization
- Dataset: `iirc_data_2006` (Woo et al.): 2cm insulated inclusion in 20cm tank
  - using interpolated data on active electrodes (**H.**, Inverse Problems 2015)

Monotonicity-based Uniqueness and Lipschitz-stability

Uniqueness

Monotonicity & localized potentials yield uniqueness results:

- Non-linear Calderón problem: (Kohn/Vogelius 1985, **H.**/Seo 2010)
  If $\sigma_1 \in L^\infty_+(\Omega)$ fulfills (UCP) and $\sigma_2 - \sigma_1$ is pcw. analytic then

$$\Lambda(\sigma_1) - \Lambda(\sigma_2) \quad \text{implies} \quad \sigma_1 = \sigma_2.$$

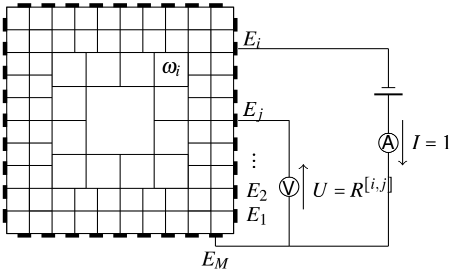
- Linearized Calderón problem: (**H.**/Seo 2010)
  If $\sigma_1 \in L^\infty_+(\Omega)$ fulfills (UCP) and $\kappa \in L^\infty(\Omega)$ is pcw. analytic then

$$\Lambda'(\sigma_1)\kappa = 0 \quad \text{implies} \quad \kappa = 0.$$

- Linearized & discretized Calderón problem: (Lechleiter/Rieder 2008)
  With enough electrodes, the linearized Calderón problem with CEM is uniquely solvable in fin.-dim. subspaces of pcw. analytic functions (e.g., pcw. polynomials of fixed degree on fixed partition).

Complete Electrode Model

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega$$
$$u|_{E_m} + z\sigma \partial_\nu u|_{E_m} = \text{const.} =: U_m$$
$$\int_{E_m} \sigma \partial_\nu u|_{E_m} \, \mathrm{d}s = J_m$$
$$\sigma \partial_\nu u = 0 \quad \text{else}$$



Current-to-Voltage operator

$$R_M(\sigma): \mathbb{R}^M_\diamond \to \mathbb{R}^M_\diamond, \quad (J_1, \ldots, J_M) \mapsto (U_1, \ldots, U_M).$$

What constraints on σ can make the inverse problem
$R_M(\sigma) \mapsto \sigma$ well-posed?

Uniqueness and Lipschitz-stability for fixed resolution

Assumptions:

- Increasing number of electrodes fulfilling Hyvönen conditions
- $\mathcal{F}$: finite-dimensional subset of pcw.-analytic functions

  (e.g., pcw. constant on fixed a-priori known partition)
- Known background conductivity:
  $\exists U$ nbr.hood of $\partial\Omega$, $\sigma_0 \in C^\infty$, so that $\sigma|_U = \sigma_0|_U$ for all $\sigma \in \mathcal{F}$
- A-prior known bounds

$$\mathcal{F}_{[a,b]} := \{\sigma \in \mathcal{F} : \ a \leq \sigma(x) \leq b \text{ for all } x \in \Omega\}$$

Theorem. (**H**, submitted) $\exists N \in \mathbb{N}, \ c > 0$:

$$\|R_M(\sigma_1) - R_M(\sigma_2)\|_{\mathcal{L}(\mathbb{R}^M_\diamond)} \geq c \, \|\sigma_1 - \sigma_2\|_{L^\infty(\Omega)} \quad \forall \sigma_1, \sigma_2 \in \mathcal{F}_{[a,b]}, M \geq N.$$

Proof (main ideas)

- Monotonicity (**H**/Ullrich, 2015)

$$\langle \big(R'(\sigma_2)(\sigma_1-\sigma_2)\big)J,J\rangle_M = \int_\Omega (\sigma_2-\sigma_1)|\nabla u_{\sigma_2}^{(J)}|^2 \,\mathrm{d}x$$
$$\leq \langle (R_M(\sigma_1)-R_M(\sigma_2))J,J\rangle_M .$$

$\rightsquigarrow$ Lower bound on Lipschitz stability

$$\|R_M(\sigma_1)-R_M(\sigma_2)\| \geq \|\sigma_1-\sigma_2\| \inf_{\substack{(\tau_1,\tau_2,\kappa)\\ \in\mathcal{F}_{[a,b]}\times\mathcal{F}_{[a,b]}\times\mathcal{K}}} \sup_{\substack{J\in\mathbb{R}^M_\diamond\\ \|J\|=1}} f_M(\tau_1,\tau_2,\kappa,J),$$

$$f_M(\tau_1,\tau_2,\kappa,J) := \max\left\{\left\langle\big(R_M'(\tau_1)\kappa\big)J,J\right\rangle, -\left\langle\big(R_M'(\tau_2)\kappa\big)J,J\right\rangle\right\},$$

- Relation to NtD-operators, localized potentials & compactness

$$\inf_{\substack{(\tau_1,\tau_2,\kappa)\\ \in\mathcal{F}_{[a,b]}\times\mathcal{F}_{[a,b]}\times\mathcal{K}}} \sup_{\substack{J\in\mathbb{R}^M_\diamond\\ \|J\|=1}} f_M(\tau_1,\tau_2,\kappa,J) > 0$$

$\square$

Conclusions

Ikehata-Kang-Seo-Sheen Monotonicity yields

- fundamental relation between measurements and unknowns,
- convergent inclusion detection methods,
- rigorous regularizers for residuum-based methods,
- theoretical uniqueness and Lipschitz stability results.

Approach can be extended

- to partial boundary data, independently of dimension $n \geq 2$,
- to stochastic settings,
- at least partially to closely related problems
  (diffuse optical tomography, magnetostatics, inverse scattering, eddy-current equations, p-Laplacian, fractional diffusion, . . .)