The Monotonicity Method for the Helmholtz equation

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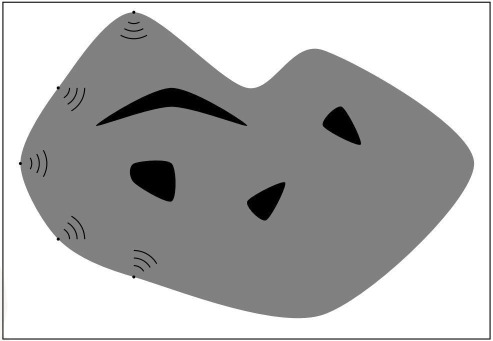
Scattering in a bounded domain



- Excite time-harmonic pressure wave in a bounded domain
- Aim: Detect defects/anomalies from scattering response
- Applications: Acoustic/EM tomography, non-destructive testing

Helmholtz equation

- Time-harmonic wave in bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$

$$\left(\Delta + k^2 q\right) u = 0 \quad \text{in } \Omega \qquad (1)$$

($k > 0$: non-resonant wavenumber, $q \in L^\infty(\Omega)$: sound speed,
$u \in H^1(\Omega)$: acoustic pressure)

- Idealized boundary mesurements: Neumann-to-Dirichlet map

$$\Lambda(q):\ L^2(\Sigma) \to L^2(\Sigma), \quad g \mapsto u|_\Sigma,$$

where u solves (1) with $\partial_\nu u|_{\partial\Omega} = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else.} \end{cases}$

($\Sigma \subseteq \partial\Omega$: open boundary part)

Can we recover q from $\Lambda(q)$?

Inversion methods

- Linearization (Born / single scattering) & Iteration
  - generic, popular, but no convergence theory
- Linear Sampling Methods / Factorization Methods
  *(Scattering: Colton, Kirsch, Cakoni, Haddar, Arens, Lechleiter, Griesmaier, . . .
  EIT: Hanke, Brühl, Hyvönen, **H.**, Seo, . . .)*
  - rigorous for exact data, yields uniqueness results
  - non-intuitive criterion (range/infinity tests)
  - no convergence theory for noisy data, needs definiteness
- Monotonicity Method (for EIT)
  *(Tamburrino, Rubinacci, **H.**, Ullrich, Mach, Garde, . . .)*
  - rigorous theory (based on FM), yields uniqueness results
  - simple, convergent for noisy data, can treat indefinite case
  - can be combined with linearization approach

This talk: Extend monotonicity method to Helmholtz equation

Monotonicity Method (for simple test case in EIT)

- EIT: Detect $\sigma \in L^\infty_+(\Omega)$ in $\nabla \cdot (\sigma \nabla u) = 0$ from NtD $\Lambda(\sigma)$
- Inclusion detection: $\sigma = 1 + \chi_D$, D open, $\Omega \setminus \overline{D}$ connected
- Monotonicity:

$$\sigma_1 \leq \sigma_2 \quad \Longrightarrow \quad \Lambda(\sigma_1) \geq \Lambda(\sigma_2)$$

(i.e., $\Lambda(\sigma_1) - \Lambda(\sigma_2)$ has **no** negative eigenvalues)

- Monotonicity for inclusion detection:
  *(„$\Longrightarrow$": Tamburrino/Rubinacci 2002, „$\Longleftarrow$" & Linearization: **H.**/Ullrich 2013)*

$$\begin{aligned} B \subseteq D \quad &\Longleftrightarrow \quad \Lambda(1+\chi_B) \geq \Lambda(1+\chi_D) \\ &\Longleftrightarrow \quad \Lambda(1) + \tfrac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(1+\chi_D) \end{aligned}$$

Inclusion can be found by testing several small balls B

Monotonicity Method for Helmholtz (simple version)

- Helmholtz: Detect $q \in L^\infty(\Omega)$ in $(\Delta + k^2 q)u = 0$ from NtD $\Lambda(q)$
- Scatterer detection: $q = 1 + \chi_D$, D open, $\Omega \setminus \overline{D}$ connected
- Monotonicity:

$$q_1 \leq q_2 \quad \Longrightarrow \quad \Lambda(q_1) \leq_{\text{fin}} \Lambda(q_2)$$

(i.e., $\Lambda(\sigma_2) - \Lambda(\sigma_1)$ has **only finitely many** negative eigenvalues)

- Monotonicity for inverse scattering:

$$\begin{aligned} B \subseteq D \quad &\Longleftrightarrow \quad \Lambda(1+\chi_B) \leq_{\text{fin}} \Lambda(1+\chi_D) \\ &\Longleftrightarrow \quad \Lambda(1) + \Lambda'(1)\chi_B \leq_{\text{fin}} \Lambda(1+\chi_D) \end{aligned}$$

Scatterer can be found by testing several small balls B

Next slides: Full results under general assumptions

Monotonicity for Helmholtz

Theorem. (**H.**/Pohjola/Salo, submitted)

Let $q_1, q_2 \in L^\infty(\Omega)$, $k > 0$ no resonance. Then

$$q_1 \leq q_2 \quad \text{implies} \quad \Lambda(q_1) \leq_{d(q_2)} \Lambda(q_2),$$

(i.e., $\Lambda(\sigma_2) - \Lambda(\sigma_1)$ has **less than** $d(q_2)$ negative eigenvalues)

- $d(q_2)$=no. of positive Neumann EVs of $\Delta + k^2 q$ (always finite)

Larger sound speed leads to larger NtD-measurements

(in the sense of a modified Loewner order)

Local uniqueness for Helmholtz

Theorem. (**H.**/Pohjola/Salo, submitted) Let

- $q_1, q_2 \in L^\infty(\Omega)$, $k > 0$ no resonance,
- $O \subseteq \overline{\Omega}$ rel. open set connected to Σ with $q_1|_O \leq q_2|_O$.

Then

$$q_1|_O \not\equiv q_2|_O \quad \text{implies} \quad \Lambda(q_1) \not\geq_{\text{fin}} \Lambda(q_2).$$

Deviation in sound speed can be detected

(from eigenvalues in NtD difference)

Scatterer detection (definite case)

$\Lambda(1)$: NtD for homogeneous sound spped

$\Lambda(q)$: NtD for unknown sound speed ($q \in L^\infty(\Omega)$, $k > 0$ no resonance)

$D \subseteq \Omega$: unknown scatterer (open, $\Omega \setminus \overline{D}$ connected)

T_B: test operator for open $B \subseteq \Omega$ ($\int_\Sigma g T_B h := \int_B k^2 u_1^g u_1^h \, \mathrm{d}x$)

Theorem. (**H.**/Pohjola/Salo, submitted)
Let $1 \leq q_{\text{min}} \leq q(x) \leq q_{\text{max}}$ for all $x \in D$ (a.e.), then

$B \subseteq D$ implies $\alpha T_B \leq_{d(q_{\text{max}})} \Lambda(q) - \Lambda(1)$ for all $\alpha \leq q_{\text{min}} - 1$,

$B \not\subseteq D$ implies $\alpha T_B \not\leq_{\text{fin}} \Lambda(q) - \Lambda(1)$ for all $\alpha > 0$.

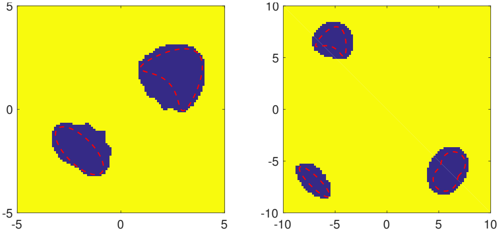
(Similar result holds for $q_{\text{min}} \leq q(x) \leq q_{\text{max}} < 1$)

Scatterer can be localized by monotonicity tests

Remarks and Extensions

- Monotonicity tests require no forward solutions (only for $q_0 \equiv 1$).
- Tests can be easily regularized ($\rightsquigarrow$ convergence for noisy data)
- Extensions possible for background sound speed $q_0 \not\equiv 1$
- Extensions possible for $\Omega \setminus \overline{D}$ not connected

  (using concept of inner and outer support)
- Extension possible for indefinite case (by shrinking large test domain)
- Extension to far-field scattering: (Griesmaier/**H.**, submitted)



Proofs (main ideas): Well-posedness

- Standard variational formulation: $u \in H^1(\Omega)$ solves

$$(\Delta + k^2 q) u = 0 \quad \text{in } \Omega, \qquad \partial_\nu u|_{\partial\Omega} = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else,} \end{cases}$$

if and only if

$$b(u,v) := \int_\Omega \left(\nabla u \cdot \nabla v - k^2 q u v \right) \mathrm{d}x = \int_\Sigma g v|_\Sigma \mathrm{d}s$$

- $b(\cdot,\cdot)$ is coercive plus compact depending analytically on $k \in \mathbb{C}$
- Analytic Fredholm theory $\rightsquigarrow$ Unique solvability

  (except for discrete set of resonance frequencies)

Proofs (main ideas): Monotonicity

- From the variational formulation one obtains

$$\int_\Sigma g\,(\Lambda(q_2)-\Lambda(q_1))\,g\,\mathrm{d}s + \int_\Omega k^2(q_1-q_2)|u_1^{(g)}|^2\,\mathrm{d}x$$
$$= \int_\Omega \left(\left|\nabla (u_2^{(g)}-u_1^{(g)})\right|^2 - k^2 q_2 |u_2^{(g)}-u_1^{(g)}|^2 \right) \mathrm{d}x.$$

- Right hand side is coercive plus compact

$\rightsquigarrow$ Right hand side is non-negative is space of finite codimension

$\rightsquigarrow$ Monotonicity inequality

$$\int_\Sigma g\,(\Lambda(q_2)-\Lambda(q_1))\,g\,\mathrm{d}s \geq_{\text{fin}} \int_\Omega k^2(q_2-q_1)|u_1^{(g)}|^2\,\mathrm{d}x$$

- Converse monotonicity by controlling $u_1^{(g)}|_D$ on subset $D\subset\Omega$

Proofs (main ideas): Localized potentials

Localized potentials: Control $u^{(g)}|_D$ on subset $D \subseteq \Omega$

- Neumann-to-Solution-operator:

$$L_D : L^2(\Sigma) \to L^2(D), \quad g \mapsto u^{(g)}|_D$$

- $L_D^* : L^2(D) \to L^2(\Sigma)$: Source-to-Dirichlet-operator
- Unique continuation: For *"different"* subsets $B, D \subseteq \Omega$

$$\mathcal{R}(L_D^*) \cap \mathcal{R}(L_B^*) = 0$$

- Duality argument: $\exists g_n \in L^2(\Sigma)$:

$$\|u^{(g_n)}|_D\| = \|L_D g_n\| \to \infty \quad \text{and} \quad \|u^{(g_n)}|_B\| = \|L_B g_n\| \to 0.$$

- $\dim \mathcal{R}(L_D^*), \dim \mathcal{R}(L_B^*) = \infty$

  $\rightsquigarrow$ g_n can be chosen from space with finite codimension

Summary

Modified Loewner order for compact selfadjoint operators:

$$A \leq_d B \quad :\Longleftrightarrow \quad B - A \text{ has less than } d \text{ negative EVs}$$

Monotonicity and converse monotonicity for Helmholtz equation:

- Larger sound speed implies larger NtD measurements.
- Larger NtD implies that there is no boundary neighbourhood where sound speed is smaller.

Monotonicity approach yields

- Local uniqueness result for Helmholtz equation
- Simple but rigorously convergent scatterer detection algorithm