of $\boldsymbol{\alpha}$ in left and right bases determined by the positions of the receiver array, the source and the scatterer.

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The monotonicity method for inverse scattering

Bastian Harrach

(joint work with Mikko Salo, Valter Pohjola)

We consider the problem of determining the support of an unknown scatterer in a bounded domain from knowledge of the associated Neumann-Dirichlet-operator for the Helmholtz equation. We show that the support can be uniquely reconstructed from operator comparisons in the sense of operator definiteness up to finitely many eigenvalues. This extends previous works on coercive equations such as EIT [4] to coercive-plus-compact equations, and yields a constructive characterization of scatterers, that is numerically stable in the sense that is allows convergent implementations for noisy data. The results that we present herein have to be considered work-in-progress, and we only sketch the main ideas for a sample case.

The setting. Let

$$\Lambda_0 : \; L^2(\partial\Omega) \to L^2(\partial\Omega), \quad g \mapsto u_0^{(g)}|_{\partial\Omega} \tag{1}$$

be the Neumann-Dirichlet-operator for the homogeneous Helmholtz equation in a bounded domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, with smooth boundary $\partial\Omega$, i.e. $u_0^{(g)} \in H^1(\Omega)$ solves

$$\Delta u_0^{(g)}(x) + k^2 u_0^{(g)}(x) = 0 \quad \text{in } \Omega, \quad \partial_\nu u_0^{(g)}|_{\partial\Omega} = g. \tag{2}$$

We also consider the case where the domain contains an open scatterer $D \subset \Omega$ with $\overline{D} \subset \Omega$ and $\Omega \setminus \overline{D}$ is connected. We assume that the refractive index in D is real-valued and strictly larger than the background, so that the scattering coefficient is given by $1+q(x)$, where $q \in L^\infty(\Omega)$ is assumed to fulfill that $q(x) = 0$ (a.e.) outside D and

$$0 < q_{\min} \leq q(x) \leq q_{\max} \quad \text{for all } x \in D \text{ (a.e.)}$$

Then the scattering field $u^{(g)} \in H^1(\Omega)$ solves

$$\Delta u_q^{(g)}(x) + k^2(1+q(x))u_q^{(g)}(x) = 0 \quad \text{in } \Omega, \quad \partial_\nu u_q^{(g)}|_{\partial\Omega} = g, \tag{3}$$

and the corresponding Neumann-Dirichlet-operator is denoted by

$$\Lambda_q:\ L^2(\partial\Omega) \to L^2(\partial\Omega), \quad g \mapsto u_q^{(g)}|_{\partial\Omega}. \tag{4}$$

We assume that k^2 is not a resonance, neither for the homogeneous nor for the inhomogeneous problem, so that both, (2) and (4), are uniquely solvable for all $g \in L^2(\partial\Omega)$ and the Neumann-Dirichlet-operators are well-defined.

The main result. Our main result is a constructive proof that D is uniquely determined from comparing Λ_q with Λ_0. For an open set B (e.g., a small ball), we introduce the self-adjoint compact test operator

$$T_B:\ L^2(\partial\Omega) \to L^2(\partial\Omega), \quad \int_{\partial\Omega} g T_B h := \int_B k^2 u_0^{(g)} u_0^{(h)}.$$

Theorem 1. *There exists a number $d_{max} \in \mathbb{N}$ such that*

(a) if $B \subseteq D$ then

$$\alpha T_B \leq_{d_{max}} \Lambda(q) - \Lambda(0) \quad \text{for all } \alpha \leq q_{min}.$$

(b) if $B \not\subseteq D$ then, for all $\alpha > 0$, $\Lambda(q) - \Lambda(0) - \alpha T_B$ has infinitely many negative eigenvalues,

where $\alpha T_B \leq_{d_{max}} \Lambda(q) - \Lambda(0)$ denotes that the difference $\Lambda(q) - \Lambda(0) - \alpha T_B$ has at most d_{max} negative eigenvalues. The number d_{max} only depends on q_{max} and can be calculated without knowledge of D.

Proof of the main result. The proof of theorem 1 follows the approach in [4] (see also [2, 3, 5] for uniqueness proofs based on this approach) and combines a monotonicity estimate with the idea of localized potentials from [1].

Lemma 2 (Monotonicity)**.** *There exists a number $d_{max} \in \mathbb{N}_0$ such that*

$$\int_{\partial\Omega} g\,(\Lambda(q) - \Lambda(0))\, g \geq_{d_{max}} \int_\Omega k^2 q |u_0^{(g)}|^2$$

Proof. From the variational formulations of (1) and (3) one obtains that

$$\begin{aligned}
&\int_{\partial\Omega} g\,(\Lambda(q) - \Lambda(0))\, g - \int_\Omega k^2 q |u_0^{(g)}|^2 \\
&\geq \int_\Omega \left(\left|\nabla(u_q^{(g)} - u_0^{(g)})\right|^2 - k^2(1+q_{\max})|u_q^{(g)} - u_0^{(g)}|^2 \right) \\
&= \langle \left(I - (1 + k^2(1+q_{\max}))K\right)(u_q^{(g)} - u_0^{(g)}), u_q^{(g)} - u_0^{(g)} \rangle_{H^1(\Omega)},
\end{aligned}$$

where I is the identity on $H^1(\Omega)$ and

$$K:\ H^1(\Omega) \to H^1(\Omega), \quad \langle Ku, v\rangle_{H^1(\Omega)} := \int_\Omega uv,$$

is compact. The assertion now follows from the fact that $I - (1 + k^2(1+q_{\max}))K$ can only have a finite number $d_{\max} \in \mathbb{N}_0$ of negative eigenvalues. $\square$

Lemma 3 (Localized potentials)**.** *If $B \subseteq \Omega$ is open and $B \not\subseteq D$ then for each finite dimensional subspace $V \subseteq L^2(\partial\Omega)$,*

$$\exists (g_k)_{k\in\mathbb{N}} \subseteq V^\perp : \quad \int_B k^2 |u_0^{(g_k)}|^2 \to \infty \quad \textit{but} \quad \int_D k^2 |u_0^{(g_k)}|^2 \to 0. \tag{5}$$

Moreover, for this sequence also $\int_D k^2 |u_q^{(g_k)}|^2 \to 0$.

Proof. By shrinking B, we can assume w.l.o.g. that $\overline{B} \subseteq \Omega$, $\overline{B} \cap \overline{D} = \emptyset$ and that $\Omega \setminus (\overline{B} \cup \overline{D})$ is connected. We then argue by contradiction, and assume that (5) is not true. Then, with the Neumann-to-Solution operators

$$L_D : \ L^2(\partial\Omega) \to L^2(D), \quad g \mapsto u_0^{(g)}|_D,$$
$$L_B : \ L^2(\partial\Omega) \to L^2(B), \quad g \mapsto u_0^{(g)}|_B,$$

there would exist a constant $C > 0$ such that

$$\|L_B g\| \le C \|L_D g\| \quad \text{for all } g \in V^\perp.$$

This would yield that there exists a self-adjoint compact F with $\dim(F) < \infty$ and

$$\|L_B g\|^2 \le C^2 \|L_D g\|^2 + \|F g\|^2 \quad \text{for all } g \in L^2(\partial\Omega).$$

Using a powerful relation between norms of operator evaluations and the ranges of their adjoints [1, Lemma 2.5]), this would imply that

$$\mathcal{R}(L_B^*) \subseteq \mathcal{R}(L_D^*) + \mathcal{R}(F). \tag{6}$$

However, the adjoints L_D^* and L_B^* can be characterized as Source-to-Dirichlet operators, and using a unique continuation argument as in part (b) of the proof of theorem 3.6 in [4], one can show that

$$\mathcal{R}(L_D^*) \cap \mathcal{R}(L_B^*) = \{0\},$$

and that $\mathcal{R}(L_D^*), \mathcal{R}(L_B^*) \subseteq L^2(\partial\Omega)$ are both dense and thus infinite-dimensional. By a dimension argument, it thus follows that (6) cannot be true. This proves (5).

Defining $\tilde{L}_D : \ L^2(\partial\Omega) \to L^2(D)$, $g \mapsto u_q^{(g)}|_D$ and using that $q = 0$ outside of D, one can show that $\mathcal{R}(\tilde{L}_D^*) = \mathcal{R}(L_D^*)$. Hence, the additional assertion $\int_D k^2 |u_q^{(g_k)}|^2 \to 0$ follows by the same arguments. □

Proof of theorem 1. If $B \subseteq D$ and $\alpha \le q_{\min}$ then lemma 2 yields that

$$\alpha \int_{\partial\Omega} g T_B g = \alpha \int_B k^2 |u_0^{(g)}|^2 \le \int_\Omega k^2 q |u_0^{(g)}|^2 \le_{d_{\max}} \int_{\partial\Omega} g\,(\Lambda(q) - \Lambda(0))\,g,$$

which shows (a). Interchanging u_q and u_0 in lemma 2, it also follows that

$$\int_{\partial\Omega} g\,(\Lambda(q) - \Lambda(0))\,g \le_{d_{\max}} \int_\Omega k^2 q |u_q^{(g)}|^2$$

Hence, if $B \not\subseteq D$, but $\Lambda(q) - \Lambda(0) \ge_{\text{fin}} \alpha T_B$, then this would imply that

$$\alpha \int_B k^2 |u_0^{(g)}|^2 \le \int_\Omega k^2 q |u_q^{(g)}|^2 = \int_D k^2 q_{\max} |u_q^{(g)}|^2$$

holds for all $g \in V^\perp$ with some finite-dimensional space $V \subset L^2(\partial\Omega)$. But this contradicts lemma 3 and thus proves (b). $\square$

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Stekloff Eigenvalues in Inverse Scattering

Shixu Meng

(joint work with Fioralba Cakoni, David Colton, Peter Monk)

We consider a problem in non-destructive testing in which small changes in the (possibly complex valued) refractive index $n(x)$ of an inhomogeneous medium of compact support are to be determined from changes in measured far field data due to incident plane waves.

It is known that transmission eigenvalues can be determined from the measured scattering data and carry information about the refractive index of non-absorbing media [3]. However the use of transmission eigenvalues in nondestructive testing has two major drawbacks. The first drawback is that in general only the first transmission eigenvalue can be accurately determined from the measured data [2] and the determination of this eigenvalue means that the frequency of the interrogating wave must be varied in a frequency range around this eigenvalue. In particular, multi-frequency data must be used in an a priori determined frequency range. This also requires the medium to be non-dispersive. The second drawback is that only real transmission eigenvalues can be conveniently determined from the measured scattering data which means that transmission eigenvalues cannot be used for the non-destructive testing of inhomogeneous absorbing media.

To overcome these difficulties, we consider a modified far field operator $\mathcal{F}$ whose kernel is the difference of the measured far field pattern due to the scattering object and the far field pattern of an auxiliary scattering problem with the Stekloff boundary condition imposed on the boundary of a domain B where B is either the support of the scattering object or a ball containing the scattering object in its interior. It is shown that $\mathcal{F}$ can be used to determine the Stekloff eigenvalues corresponding to B where if $B \neq D$ the refractive index is set equal to one in $B \setminus \overline{D}$. For fixed k, $\lambda := \lambda(k) \in \mathbb{C}$ is called a Stekloff eigenvalue if there exists a