Inverse problems and medical imaging

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Introduction to inverse problems

Laplace's demon

Pierre Simon Laplace (1814):

”An intellect which … would know all forces … and all positions of all items,

if this intellect were also vast enough to submit these data to analysis …

for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.”

Computational Science:

If we know all necessary parameters, then we can numerically predict the outcome of an experiment (by solving mathematical formulas).

Goals:

- Prediction
- Optimization
- Inversion/Identification

Generic simulation problem:

Given input x calculate outcome $y = F(x)$.

$$
\begin{aligned}
x \in X&: \quad \text{parameters / input}\\
y \in Y&: \quad \text{outcome / measurements}\\
F: X \to Y&: \quad \text{functional relation / model}
\end{aligned}
$$

Goals:

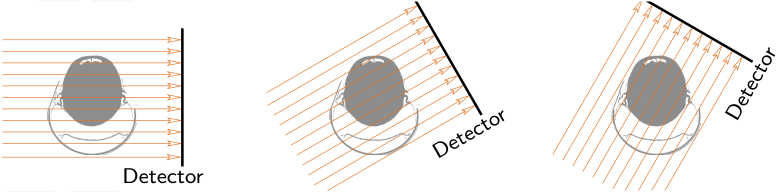
- Prediction: Given x, calculate $y = F(x)$.
- Optimization: Find x, such that $F(x)$ is optimal.
- Inversion/Identification: Given $F(x)$, calculate x.

Example: X-ray computerized tomography (CT)

Nobel Prize in Physiology or Medicine 1979:
Allan M. Cormack and Godfrey N. Hounsfield

(Photos: Copyright ©The Nobel Foundation)

Idea: Take x-ray images from several directions

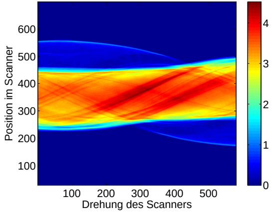


Computerized tomography (CT)

(Image: Hanke-Bourgeois, Grundlagen der Numerischen Mathematik und des Wiss. Rechnens, Teubner 2002)

Image

$\overset{F}{\longmapsto}$



Measurements

Direct problem: Simulate/predict the measurements

(from knowledge of the interior density distribution)

Given x *calculate* $F(x) = y$*!*

Inverse problem: Reconstruct/image the interior distribution

(from taking x-ray measurements)

Given y *solve* $F(x) = y$*!*

Computerized tomography

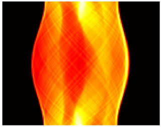
- CT forward operator $F: x \mapsto y$ is linear

$\rightsquigarrow$ Evaluation of F is simple matrix vector multiplication

(after discretizing image and measurements as long vectors)

Simple low resolution example:

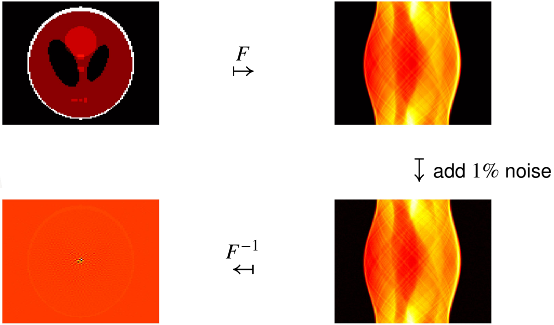
$$\overset{F}{\mapsto} \qquad \overset{F^{-1}}{\mapsfrom}$$



Problem: Matrix F invertible, but $\|F^{-1}\|$ very large.

Ill-posedness

- In the continuous case: F^{-1} not continuous
- After discretization: $\|F^{-1}\|$ very large



Are stable reconstructions impossible?

Ill-posedness

Generic linear ill-posed inverse problem

- $F: X \to Y$ bounded and linear, X, Y Hilbert spaces,
- F injective, F^{-1} not continuous,
- True solution and noise-free measurements: $F\hat{x} = \hat{y}$,
- Real measurements: y^δ with $\|y^\delta - \hat{y}\| \leq \delta$

$$F^{-1}y^\delta \not\to F^{-1}\hat{y} = \hat{x} \quad \text{for} \quad \delta \to 0.$$

Even the smallest noise may corrupt the reconstructions.

Regularization

Generic linear Tikhonov regularization

$$R_\alpha = (F^*F + \alpha I)^{-1} F^*$$

$\rightsquigarrow$ R_α continuous, $x = R_\alpha y^\delta$ minimizes

$$\|Fx - y^\delta\|^2 + \alpha \|x\|^2 \to \text{min!}$$

Theorem. Choose $\alpha := \delta$. Then for $\delta \to 0$,

$$R_\delta y^\delta \to F^{-1}\hat{y}.$$

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Proof. Show that $\|R_\alpha\| \leq \frac{1}{\sqrt{\alpha}}$ and apply

$$\|R_\alpha y^\delta - F^{-1}\hat{y}\| \leq \underbrace{\|R_\alpha(y^\delta - \hat{y})\|}_{\leq \|R_\alpha\|\,\delta} + \underbrace{\|R_\alpha \hat{y} - F^{-1}y\|}_{\to 0 \text{ for } \alpha \to 0}.$$

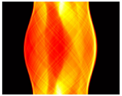
Inexact but continuous reconstruction (regularization)

\+ Information on measurement noise (parameter choice rule)

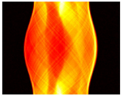
= Convergence

Example ($\delta = 1\%$)

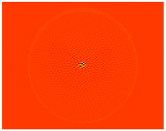
$\hat{x}$



$\hat{y} = F\hat{x}$



y^δ

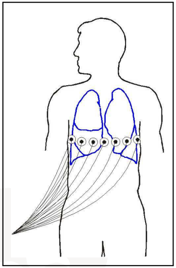


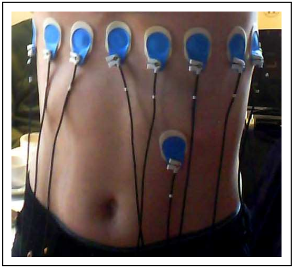
$F^{-1}y^\delta$

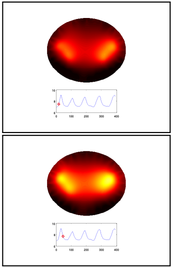
$(F^*F + \delta I)^{-1}F^*y^\delta$

Electrical impedance tomography

Electrical impedance tomography (EIT)







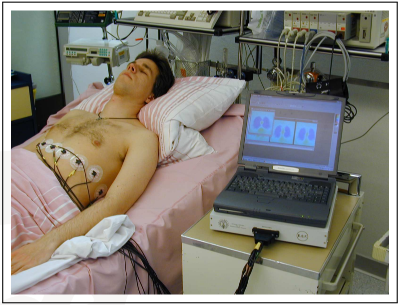
- Apply electric currents on subject's boundary
- Measure necessary voltages

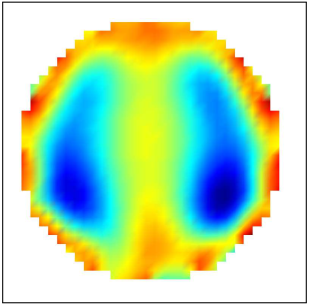
$\rightsquigarrow$ Reconstruct conductivity inside subject.

Images from BMBF-project on EIT

(Hanke, Kirsch, Kress, Hahn, Weller, Schilcher, 2007-2010)

MF-System Goe-MF II





Electric current strength: $5 - 500\mathrm{mA}_{\mathrm{rms}}$, 44 images/second,
CE certified by Viasys Healthcare, approved for clinical research

Mathematical Model

- Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega \qquad \text{(EIT)}$$

$\Omega \subset \mathbb{R}^n$: imaged body, $n \geq 2$
$\sigma(x)$: conductivity
$u(x)$: electrical potential

- Idealistic model for boundary meas. (continuum model):

$\sigma \partial_\nu u(x)|_{\partial\Omega}$: applied electric current
$u(x)|_{\partial\Omega}$: measured boundary voltage (potential)

- Neumann-to-Dirichlet-Operator:

$$\Lambda(\sigma): L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (EIT) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Electrical impedance tomography

Inverse problem of EIT: Recover σ from $\Lambda(\sigma)$

Challenges:

- Uniqueness
  - Is σ uniquely determined from "perfect data" $\Lambda(\sigma)$?
- Non-linearity and ill-posedness
  - Reconstruction algorithms to determine σ from $\Lambda(\sigma)$?
  - Local/global convergence results for noisy data $\Lambda^{\delta}_{\text{meas}} \approx \Lambda(\sigma)$?
- Realistic data
  - What can we recover from real measurements?

    (fixed number of electrodes, realistic electrode models, . . .)
  - Measurement and modelling errors? Resolution?

Inversion of $\sigma \mapsto \Lambda(\sigma) = \Lambda_{\text{meas}}$?

Generic solvers for non-linear inverse problems:

- Linearize and regularize:

  $$\Lambda_{\text{meas}} = \Lambda(\sigma) \approx \Lambda(\sigma_0) + \Lambda'(\sigma_0)(\sigma - \sigma_0).$$

  σ_0: Initial guess or reference state (e.g. exhaled state)

  $\rightsquigarrow$ Linear inverse problem for σ

  (Solve, e.g., using linear Tikhonov regul., repeat for Newton-type algorithm.)

- Regularize and linearize:

  E.g., minimize non-linear Tikhonov functional

  $$\|\Lambda_{\text{meas}} - \Lambda(\sigma)\|^2 + \alpha \|\sigma - \sigma_0\|^2 \to \text{min!}$$

Very flexible, but high comput. cost and convergence unclear

Linearization and shape reconstruction

Theorem (**H.**/Seo, SIAM J. Math. Anal. 2010)
Let κ, σ, σ_0 pcw. analytic.

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \quad \Longrightarrow \quad \operatorname{supp}_{\partial\Omega}\kappa = \operatorname{supp}_{\partial\Omega}(\sigma - \sigma_0)$$

$\operatorname{supp}_{\partial\Omega}$: outer support ($=$ supp $+$ parts unreachable from $\partial\Omega$)

- Linearized EIT equation contains correct shape information
- For the shape reconstruction problem $\Lambda(\sigma) \mapsto \operatorname{supp}_{\partial\Omega}(\sigma - \sigma_0)$ fast, rigorous and globally convergent method seem possible.
- Goal: Given $\Lambda^\delta_{\text{meas}} \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, can we regularize

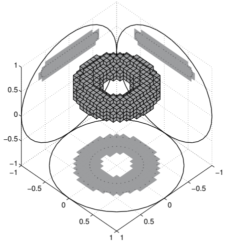
$$\|\Lambda'(\sigma_0)\kappa - \Lambda^\delta_{\text{meas}}\| \to \min!$$

so that $\operatorname{supp}_{\partial\Omega}\kappa^\delta \to \operatorname{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

Monotonicity method (for simple test example)

Theorem *(**H.**/Ullrich, SIAM J. Math. Anal. 2013)*
$\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1+\chi_B) \geq \Lambda(\sigma).$$



For faster implementation:

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1) + \tfrac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$

Shape can be reconstructed by linearized monotonicity tests.

$\rightsquigarrow$ fast, rigorous, allows globally convergent implementation

Sketch of proof

Theorem $\Omega \setminus \overline{D}$ connected, B open.

$$B \subseteq D \quad \iff \quad \Lambda(1+\chi_B) \geq \Lambda(1+\chi_D).$$

„$\Longrightarrow$": follows from monotonicity inequality:

$$\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \geq \int_{\partial\Omega} g\,(\Lambda(\sigma_0) - \Lambda(\sigma_1))\,g \geq \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)|\nabla u_0|^2$$

„$\Longleftarrow$": follows from using localized potentials in monoton. inequality.
If $B \nsubseteq D$ then there exist solutions $u_0^{(k)}$, $k \in \mathbb{N}$ with

$$\int_B \left|\nabla u_0^{(k)}\right|^2 \mathrm{d}x \to \infty \quad \text{and} \quad \int_D \left|\nabla u_0^{(k)}\right|^2 \mathrm{d}x \to 0.$$

Improving residuum-based methods

Let $\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

- Pixel partition $\Omega = \bigcup_{k=1}^{m} P_k$
- Regularized monotonicity tests

$$\beta_k^\delta \in [0, \infty] \text{ max. values s.t. } \beta_k^\delta \Lambda'(1) \chi_{P_k} \geq \Lambda_{\text{meas}}^\delta - \delta I$$

- Monotonicity-constrained residuum minimization

$$\| \Lambda'(1) \kappa^\delta - \Lambda_{\text{meas}}^\delta \|_{\text{F}} \to \text{min!}$$

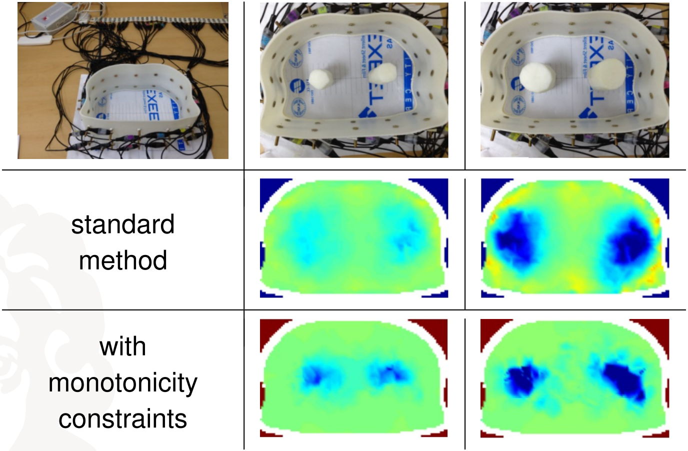
$$\text{such that} \quad \kappa^\delta|_{P_k} = \text{const.}, \ 0 \leq \kappa^\delta|_{P_k} \leq \min\{\tfrac{1}{2}, \beta_k^\delta\}$$

($\| \cdot \|_F$: Frobenius norm of Galerkin projektion to finite-dimensional space)

Theorem (**H.**/Minh, Inverse Problems 2016)

- For $\delta = 0$, there exists unique minimizer κ and
$$P_k \subseteq \operatorname{supp} \kappa \quad \Longleftrightarrow \quad P_k \subseteq \operatorname{supp}(\sigma - 1).$$
- For noisy data, minimizers κ^δ exist and $\kappa^\delta \to \kappa$ pointwise.

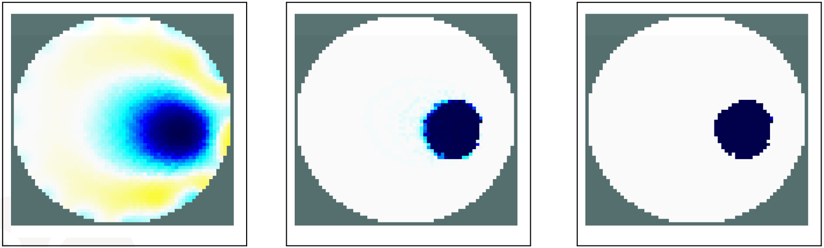
Phantom experiment



Enhancing standard methods by (heuristic) monotonicity constraints

(Zhou/**H.**/Seo, arXiv:1702.02563, 2017)

Benchmark example



standard | monoton.-constrained (Matlab quadprog) | monoton.-constrained (cvx package)

Rigorous monoton.-constrained method vs. community standard

(**H.**/Minh, Trends in Mathematics, to appear)

- EIT community standard: `inv_solve` in EIDORS
- EIDORS: `http://eidors3d.sourceforge.net` (Adler/Lionheart)
- Dataset: `iirc_data_2006` (Woo et al.): 2cm insulated inclusion in 20cm tank
  - using interpolated data on active electrodes (**H.**, Inverse Problems 2015)

Conclusions

Computational science and inverse problems

- Computational science is the core of many new advances.
- Inverse problems is the core of new medical imaging systems.

For ill-posed inverse problems

- Regularization is required for convergent algorithms.
- Regularization can also incorporate additional information
  (e.g., total variation penalization, stochastic priors, etc.)

For the non-linear ill-posed inverse problem of EIT

- Convergence of standard regularization is still unclear.
- Monotonicity-based regularization allows fast, rigorous, and globally convergent reconstruction of shape information.