Monotonicity-based regularization of inverse coefficient problems

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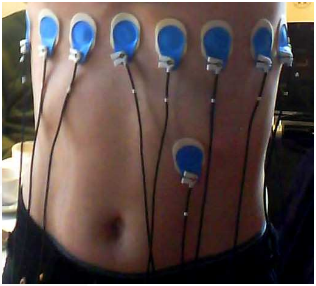
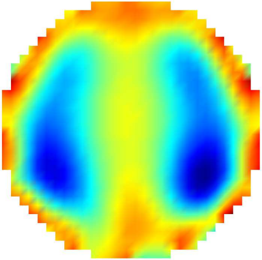
Electrical impedance tomography (EIT)





- Apply electric currents on subject's boundary
- Measure necessary voltages

$\rightsquigarrow$ Reconstruct conductivity inside subject.

Mathematical Model

Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega$$

$\Omega \subset \mathbb{R}^n$: imaged body, $n \geq 2$

$\sigma(x)$: conductivity

$u(x)$: electrical potential

Idealistic model for boundary measurements (continuum model):

$\sigma \partial_\nu u(x)|_{\partial\Omega}$: applied electric current

$u(x)|_{\partial\Omega}$: measured boundary voltage (potential)

Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves } (1)\} \,?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma) : L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Inversion of $\sigma \mapsto \Lambda(\sigma)$?

Generic optimization-based solvers for non-linear inverse problems:

- Linearize and regularize:

$$\Lambda_{\text{meas}} \approx \Lambda(\sigma) \approx \Lambda(\sigma_0) + \Lambda'(\sigma_0)(\sigma - \sigma_0).$$

  σ_0: Initial guess or reference state (e.g. exhaled state)

  $\rightsquigarrow$ Linear inverse problem for σ

  (Solve using linear regularization method, repeat for Newton-type algorithm.)

- Regularize and linearize:

  E.g., minimize non-linear Tikhonov functional

$$\left\| \Lambda_{\text{meas}} - \Lambda(\sigma) \right\|^2 + \alpha \left\| \sigma - \sigma_0 \right\|^2 \to \text{min!}$$

Advantages of generic optimization-based solvers:

- Very flexible, additional data/unknowns easily incorporated
- Problem-specific regularization can be applied

  (e.g., total variation penalization, stochastic priors, etc.)

Inversion of $\sigma \mapsto \Lambda(\sigma)$?

Problems with generic optimization-based solvers

- High computational cost
  - Evaluations of $\Lambda(\cdot)$ and $\Lambda'(\cdot)$ require PDE solutions.
  - PDE solutions too expensive for real-time imaging

- Convergence unclear

  (Validity of TCC/Scherzer-condition is a long-standing open problem for EIT.)
  - Convergence against true solution for exact meas. Λ_{meas}?

    (in the limit of infinite computation time)
  - Convergence against true solution for noisy meas. $\Lambda_{\text{meas}}^{\delta}$?

    (in the limit of $\delta \to 0$ and infinite computation time)
  - Global convergence? Resolution estimates for realistic noise?

- Influence of modelling errors
  - Evaluations of $\Lambda(\cdot)$ affected by large modelling errors

    (boundary geometry, electrode position, etc.)

Linearized methods

Popular approach in practice:

- Measure difference data $\Lambda_{\text{meas}} \approx \Lambda(\sigma) - \Lambda(\sigma_0)$.

  (e.g. $\Lambda(\sigma_0)$ measurement at exhaled state)

- Calculate $\sigma - \sigma_0$ from Λ_{meas} by single linearization step.

Standard linearized method

e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

$$\text{Solve } \Lambda'(\sigma_0)\kappa = \Lambda_{\text{meas}}, \text{ then } \kappa \approx \sigma - \sigma_0.$$

After discretization and regularization:

$$\|\mathbf{S}\kappa - \mathbf{V}\|^2 + \alpha\|\kappa\|^2 \to \text{min!}$$

$\mathbf{S}$: sensitivity matrix, $\mathbf{V}$: vector of EIT measurements.

Linearization and shape reconstruction

Theorem (**H.**/Seo, SIMA 2010)
Let κ, σ, σ_0 pcw. analytic.

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \quad \Longrightarrow \quad \mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$$

$\mathrm{supp}_{\partial\Omega}$: outer support ($= \mathrm{supp}$ + parts unreachable from $\partial\Omega$)

- Linearized EIT equation contains correct shape information.
  (in the continuous version for noise-free measurements on infinitely many electrodes)
- Practitioners use heuristic regularization of linearized EIT equ.
  (in the discretized version for noisy measurements on finitely many electrodes)

Can we minimize linearized data-fit residual with a regularization that rigorously guarantees convergence of reconstructed shapes?

Inclusion detection

Sample inclusion detection problem (for ease of presentation)

- $\sigma_0 = 1$
- $\sigma_1 = 1 + \chi_D$
- D open, $\overline{D} \subseteq \Omega$, $\Omega \setminus \overline{D}$ connected

All of the following also holds for

- σ_0 pcw. analytic,
- $\sigma_1 = \sigma_0 + \kappa\chi_D$ with $\kappa \in L^\infty_+(D)$,
- in any dimension $n \geq 2$,
- for partial boundary data on open subset $\Gamma \subseteq \partial\Omega$,

as long as σ_0 and bounds on κ are known.

Rigorous inclusion detection in EIT

- Linear Sampling Method (Scattering: Colton/Kirsch 1996):

$$z \notin D \quad \Longrightarrow \quad I(z) = \infty$$

  - Rigorously detects subset of D for exact data.

- Factorization Method (Scattering: Kirsch 1998, EIT: Brühl/Hanke 1999):

$$z \notin D \quad \Longleftrightarrow \quad I(z) = \infty.$$

  - Rigorously detects D for exact data.
  - No convergence result for noisy data $\Lambda_{\text{meas}}^{\delta} \to \Lambda(\sigma) - \Lambda(1)$.

- (Linearized) Monotonicity Method
  - Tamburrino/Rubinacci 02:

$$B \subseteq D \quad \Longrightarrow \quad \Lambda(1+\chi_B) \geq \Lambda(\sigma).$$

  - **H.**/Ullrich, SIMA 2013:

$$\begin{aligned} B \subseteq D \quad &\Longleftrightarrow \quad \Lambda(1+\chi_B) \geq \Lambda(\sigma) \\ &\Longleftrightarrow \quad \Lambda(1) + \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma). \end{aligned}$$

Monotonicity Method

H./Ullrich, SIMA 2013:

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1) + \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma)$$

- Simple to implement, no PDE solutions
- Same computational cost as FM or single linearization step
- Rigorously detects unknown shape for exact data
- Convergence for noisy data $\Lambda^{\delta}_{\text{meas}} \to \Lambda(\sigma) - \Lambda(1)$:

$$R(\Lambda^{\delta}_{\text{meas}}, \delta, B) := \begin{cases} 1 & \text{if } \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda^{\delta}_{\text{meas}} - \delta I \\ 0 & \text{else.} \end{cases}$$

Then $R(\Lambda^{\delta}_{\text{meas}}, \delta, B) \to 1$ iff $B \subseteq D$.

Monotonicity method

Quantitative, pixel-based variant of monotonicity method:

- Pixel partition $\Omega = \bigcup_{k=1}^{m} P_k$
- Quantitative monotonicity tests

$$\beta_k \in [0,\infty] \text{ max. values s.t. } \beta_k \Lambda'(1)\chi_{P_k} \geq \Lambda(\sigma) - \Lambda(1)$$
$$\beta_k^\delta \in [0,\infty] \text{ max. values s.t. } \beta_k^\delta \Lambda'(1)\chi_{P_k} \geq \Lambda_{\text{meas}}^\delta - \delta I$$

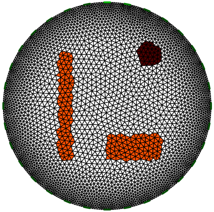
"Raise conductivity in each pixel until monotonicity test fails."

- By theory of monotonicity method:

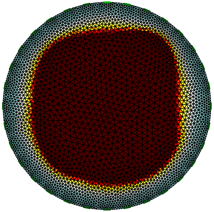
$$\beta_k^\delta \to \beta_k \quad \text{and} \quad \beta_k \text{ fulfills } \begin{cases} \beta_k = 0 & \text{if } P_k \not\subseteq D \\ \beta_k \geq \frac{1}{2} & \text{if } P_k \subseteq D \end{cases}$$

Plotting β_k^δ shows true inclusions up to pixel partition.

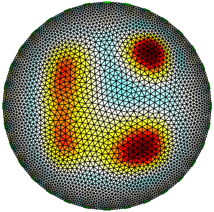
Realistic example (32 electrodes, 1% noise)



True image



Monotonicity method



Standard linearized method

- Monotonicity method rigorously converges for $\delta \to 0 \ldots$
- $\ldots$ but the heuristic standard linearized method works much better for realistic scenarios.

Can we improve the monotonicity method without loosing convergence?

Monotonicity-based regularization

- Standard linearized methods for EIT: Minimize

$$\|\Lambda'(1)\kappa - (\Lambda(\sigma) - \Lambda(1))\|^2 + \alpha \|\kappa\|^2 \to \text{min!}$$

Choice of norms heuristic. No convergence theory!

- Monotonicity-based regularization: Minimize

$$\|\Lambda'(1)\kappa - (\Lambda(\sigma) - \Lambda(1))\|_{\mathrm{F}} \to \text{min!}$$

under the constraint $\kappa|_{P_k} = \text{const.}$, $0 \leq \kappa|_{P_k} \leq \min\{\frac{1}{2}, \beta_k\}$.

($\|\cdot\|_F$: Frobenius norm of Galerkin projektion to finite-dimensional space)

Theorem (**H.**/Mach, Inverse Problems 2016)

- There exists unique minimizer $\hat{\kappa}$ and

$$P_k \subseteq \operatorname{supp} \hat{\kappa} \quad \Longleftrightarrow \quad P_k \subseteq \operatorname{supp}(\sigma - 1).$$

- Minimizer fulfills $\hat{\kappa} = \sum_{k=1}^{m} \min\{1/2, \beta_k\} \chi_{P_k}$

Monotonicity-based regularization

For noisy measurements $\Lambda^\delta_{\text{meas}} \approx \Lambda(\sigma) - \Lambda(1)$:

- Use regularized monotonicity tests

$$\beta_k^\delta \in [0,\infty] \text{ max. values s.t. } \beta_k^\delta \Lambda'(1)\chi_{P_k} \geq \Lambda^\delta_{\text{meas}} - \delta I$$

($\delta > 0$: noise level in $\mathcal{L}(L^2_\diamond(\partial\Omega))$-norm)

- Minimize

$$\|\Lambda'(1)\kappa^\delta - \Lambda^\delta_{\text{meas}}\|_{\text{F}} \to \text{min!}$$

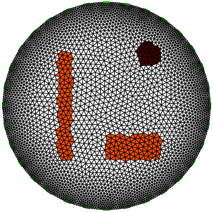
under the constraint $\kappa^\delta|_{P_k} = \text{const.}$, $0 \leq \kappa^\delta|_{P_k} \leq \min\{\frac{1}{2}, \beta_k^\delta\}$.

Theorem (**H.**/Mach, Inverse Problems 2016)

- There exist minimizers κ^δ and $\kappa^\delta \to \hat{\kappa}$ for $\delta \to 0$.

Monotonicity-regularized solutions converge against correct shape.

Realistic example (32 electrodes, 1% noise)



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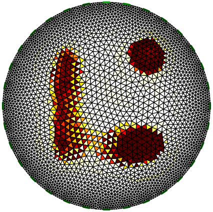
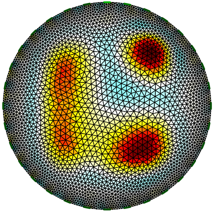


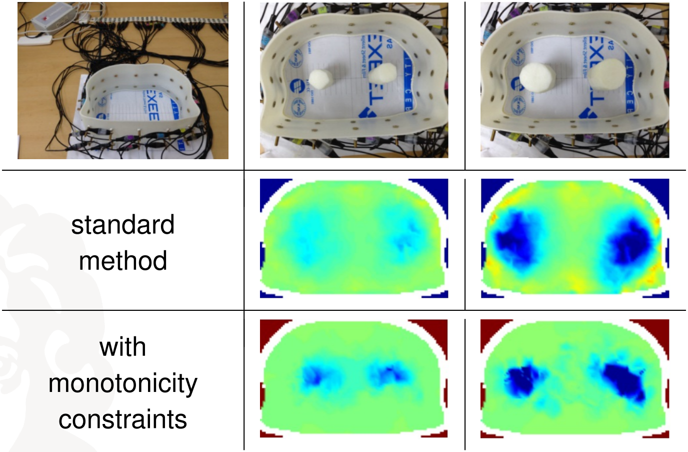
Monotonicity regularized method



Standard linearized method

- Monotonicity regularized method rigorously converges and is up to par with (outperforms?) heuristic standard linearized method.

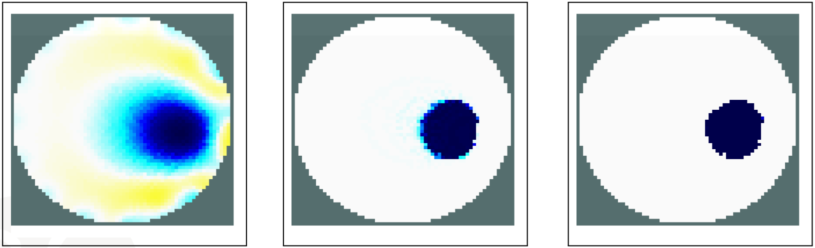
Phantom experiment



Enhancing standard methods by monotonicity-based constraints

(Zhou/**H.**/Seo, 2016)

Benchmark example



standard — monoton.-regularized (Matlab quadprog) — monoton.-regularized (cvx package)

Monotonicity-regularization vs. community standard

(**H.**/Mach, submitted)

- EIDORS: `http://eidors3d.sourceforge.net` (Adler/Lionheart)
- EIDORS standard solver: linearized method with Tikhonov regularization
- Dataset: `iirc_data_2006` (Woo et al.): 2cm insulated inclusion in 20cm tank
  - using interpolated data on active electrodes (**H.**, Inverse Problems 2015)

Conclusions

EIT is a highly ill-posed, non-linear inverse problem.

- Convergence of generic solvers unclear.
- Practitioners use single linearization step with heuristic regularization and no theoretical justification.

Monotonicity-based regularization of linearized EIT equation

- uses that shape reconstr. in EIT is (essentially) a linear problem,
- yields solutions that rigorously converge against correct shape,
- combines rigorous theory of monotonicity method with practical robustness of residuum-based methods.

Approach (monotonicity + localized potentials) can be extended

- to partial boundary data, independently of dimension $n \geq 2$
- to other linear elliptic problems *(diffuse optical tomography, magnetostatics)*
- at least partially to closely related problems

  (eddy-current equations, p-Laplacian)