

Inverse problems and medical imaging

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Introduction to inverse problems

Pierre Simon Laplace (1814):

*"An intellect which ... would know
all forces ... and all positions of all items,
if this intellect were also vast enough to
submit these data to analysis ...*

*for such an intellect nothing would be
uncertain and the future just like the past
would be present before its eyes."*



Computational Science / Simulation Technology:

If we know all necessary parameters, then we can numerically predict the outcome of an experiment (by solving mathematical formulas).

Goals:

- ▶ Prediction
- ▶ Optimization
- ▶ Inversion/Identification

Generic simulation problem:

Given input x calculate outcome $y = F(x)$.

$x \in X$: parameters / input

$y \in Y$: outcome / measurements

$F : X \rightarrow Y$: functional relation / model

Goals:

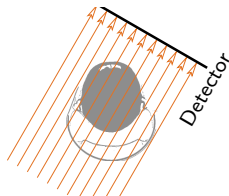
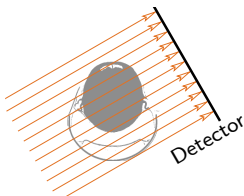
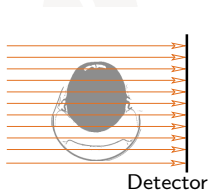
- ▶ **Prediction:** Given x , calculate $y = F(x)$.
- ▶ **Optimization:** Find x , such that $F(x)$ is optimal.
- ▶ **Inversion/Identification:** Given $F(x)$, calculate x .

Example: X-ray computerized tomography (CT)

Nobel Prize in Physiology or Medicine 1979:
Allan M. Cormack and Godfrey N. Hounsfield
(Photos: Copyright ©The Nobel Foundation)



Idea: Take x-ray images from several directions



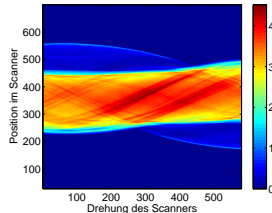
Computerized tomography (CT)

(Image: Hanke-Bourgeois, Grundlagen der Numerischen Mathematik und des Wiss. Rechnens, Teubner 2002)



Image

F



Measurements

Direct problem:

Simulate/predict the measurements

(from knowledge of the interior density distribution)

Given x calculate $F(x) = y!$

Inverse problem:

Reconstruct/image the interior distribution

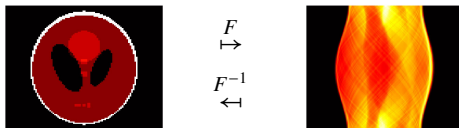
(from taking x-ray measurements)

Given y solve $F(x) = y!$

Computerized tomography

- ▶ CT forward operator $F : x \mapsto y$ is linear
- ↪ Evaluation of F is simple matrix vector multiplication
(after discretizing image and measurements as long vectors)

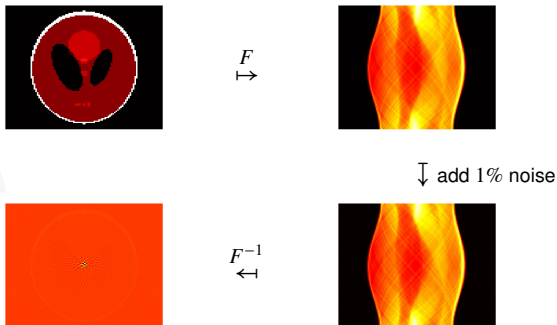
Simple low resolution example:



Problem: Matrix F invertible, but $\|F^{-1}\|$ very large.

Ill-posedness

- ▶ In the continuous case: F^{-1} not continuous
- ▶ After discretization: $\|F^{-1}\|$ very large



Are stable reconstructions impossible?

Ill-posedness

Generic linear ill-posed inverse problem

- ▶ $F : X \rightarrow Y$ bounded and linear, X, Y Hilbert spaces,
- ▶ F injective, F^{-1} not continuous,
- ▶ True solution and noise-free measurements: $F\hat{x} = \hat{y}$,
- ▶ Real measurements: y^δ with $\|y^\delta - \hat{y}\| \leq \delta$

$$F^{-1}y^\delta \not\rightarrow F^{-1}\hat{y} = \hat{x} \quad \text{for} \quad \delta \rightarrow 0.$$

Even the smallest noise may corrupt the reconstructions.

Generic linear Tikhonov regularization

$$R_{\alpha} = (F^*F + \alpha I)^{-1}F^*$$

$\leadsto R_{\alpha}$ continuous, $x = R_{\alpha}y^{\delta}$ minimizes

$$\|Fx - y^{\delta}\|^2 + \alpha \|x\|^2 \rightarrow \min!$$

Theorem. Choose $\alpha := \delta$. Then for $\delta \rightarrow 0$,

$$R_{\delta}y^{\delta} \rightarrow F^{-1}\hat{y}.$$

Regularization

Theorem. Choose $\alpha := \delta$. Then for $\delta \rightarrow 0$,

$$R_\delta y^\delta \rightarrow F^{-1} \hat{y}.$$

Proof. Show that $\|R_\alpha\| \leq \frac{1}{\sqrt{\alpha}}$ and apply

$$\|R_\alpha y^\delta - F^{-1} \hat{y}\| \leq \underbrace{\|R_\alpha(y^\delta - \hat{y})\|}_{\leq \|R_\alpha\| \delta} + \underbrace{\|R_\alpha \hat{y} - F^{-1} y\|}_{\rightarrow 0 \text{ for } \alpha \rightarrow 0}.$$

Inexact but continuous reconstruction (**regularization**)

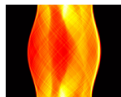
+ Information on measurement noise (**parameter choice rule**)

= Convergence

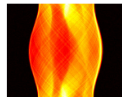
Example



\hat{x}



$\hat{y} = F \hat{x}$



y^δ



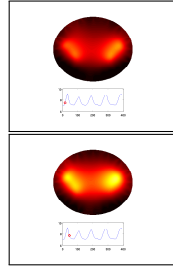
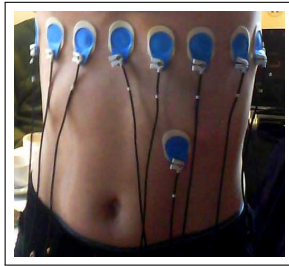
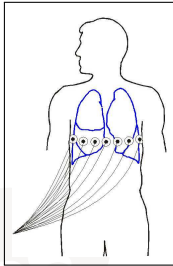
$F^{-1} y^\delta$



$(F^* F + \delta I)^{-1} F^* y^\delta$

Electrical impedance tomography

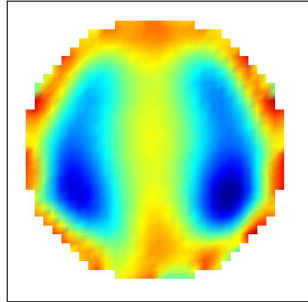
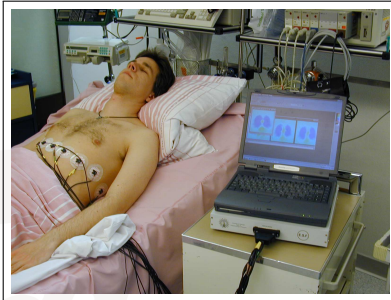
Electrical impedance tomography (EIT)



- ▶ Apply electric currents on subject's boundary
- ▶ Measure necessary voltages
- ~> Reconstruct conductivity inside subject.

Images from BMBF-project on EIT

(Hanke, Kirsch, Kress, Hahn, Weller, Schilcher, 2007-2010)



Electric current strength: $5 - 500 \text{ mA}_{\text{rms}}$, 44 images/second,
CE certified by Viasys Healthcare, approved for clinical research

- ▶ Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega \quad (\text{EIT})$$

$\Omega \subset \mathbb{R}^n$: imaged body, $n \geq 2$

$\sigma(x)$: conductivity

$u(x)$: electrical potential

- ▶ Idealistic model for boundary meas. (continuum model):

$\sigma \partial_\nu u(x)|_{\partial\Omega}$: applied electric current

$u(x)|_{\partial\Omega}$: measured boundary voltage (potential)

- ▶ Neumann-to-Dirichlet-Operator:

$$\Lambda(\sigma) : L^2_\diamond(\partial\Omega) \rightarrow L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (EIT) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Inverse problem of EIT: Recover σ from $\Lambda(\sigma)$

Challenges:

- ▶ Uniqueness
 - ▶ Is σ uniquely determined from "perfect data" $\Lambda(\sigma)$?
- ▶ Non-linearity and ill-posedness
 - ▶ Reconstruction algorithms to determine σ from $\Lambda(\sigma)$?
 - ▶ Local/global convergence results for noisy data $\Lambda_{\text{meas}}^{\delta} \approx \Lambda(\sigma)$?
- ▶ Realistic data
 - ▶ What can we recover from real measurements?
(fixed number of electrodes, realistic electrode models, ...)
 - ▶ Measurement and modelling errors? Resolution?

Inversion of $\sigma \mapsto \Lambda(\sigma) = \Lambda_{\text{meas}}$?

Generic solvers for non-linear inverse problems:

- ▶ **Linearize and regularize:**

$$\Lambda_{\text{meas}} = \Lambda(\sigma) \approx \Lambda(\sigma_0) + \Lambda'(\sigma_0)(\sigma - \sigma_0).$$

σ_0 : Initial guess or reference state (e.g. exhaled state)

↪ Linear inverse problem for σ

(Solve, e.g., using linear Tikhonov regul., repeat for Newton-type algorithm.)

- ▶ **Regularize and linearize:**

E.g., minimize non-linear Tikhonov functional

$$\|\Lambda_{\text{meas}} - \Lambda(\sigma)\|^2 + \alpha \|\sigma - \sigma_0\|^2 \rightarrow \min!$$

Very flexible, but high comput. cost and convergence unclear

Linearization and shape reconstruction

Theorem (H./Seo, SIAM J. Math. Anal. 2010)

Let κ , σ , σ_0 pcw. analytic.

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \implies \text{supp}_{\partial\Omega}\kappa = \text{supp}_{\partial\Omega}(\sigma - \sigma_0)$$

$\text{supp}_{\partial\Omega}$: outer support (= supp + parts unreachable from $\partial\Omega$)

- ▶ Linearized EIT equation contains correct shape information
- ▶ For the shape reconstruction problem $\Lambda(\sigma) \mapsto \text{supp}_{\partial\Omega}(\sigma - \sigma_0)$ fast, rigorous and globally convergent method seem possible.
- ▶ **Goal:** Given $\Lambda_{\text{meas}}^\delta \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, can we regularize

$$\|\Lambda'(\sigma_0)\kappa - \Lambda_{\text{meas}}^\delta\| \rightarrow \min!$$

so that $\text{supp}_{\partial\Omega}\kappa^\delta \rightarrow \text{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

Monotonicity method (for simple test example)

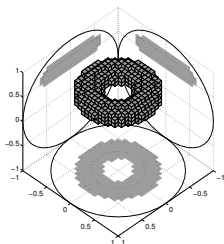
Theorem (*H./Ullrich, SIAM J. Math. Anal.* 2013)

$\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

$$B \subseteq D \iff \Lambda(1 + \chi_B) \geq \Lambda(\sigma).$$

For faster implementation:

$$B \subseteq D \iff \Lambda(1) + \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$



Shape can be reconstructed by linearized monotonicity tests.

↪ fast, rigorous, allows globally convergent implementation

Sketch of proof

Theorem $\Omega \setminus \overline{D}$ connected, B open.

$$B \subseteq D \iff \Lambda(1 + \chi_B) \geq \Lambda(1 + \chi_D).$$

„ \implies ”: follows from **monotonicity inequality**:

$$\int_{\Omega} (\sigma_1 - \sigma_0) |\nabla u_0|^2 \geq \int_{\partial\Omega} g (\Lambda(\sigma_0) - \Lambda(\sigma_1)) g \geq \int_{\Omega} \frac{\sigma_0}{\sigma_1} (\sigma_1 - \sigma_0) |\nabla u_0|^2$$

„ \impliedby ”: follows from using **localized potentials** in monoton. inequality.

If $B \not\subseteq D$ then there exist solutions $u_0^{(k)}$, $k \in \mathbb{N}$ with

$$\int_B |\nabla u_0^{(k)}|^2 dx \rightarrow \infty \quad \text{and} \quad \int_D |\nabla u_0^{(k)}|^2 dx \rightarrow 0.$$

Improving residuum-based methods

Let $\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

- ▶ Pixel partition $\Omega = \bigcup_{k=1}^m P_k$
- ▶ Regularized monotonicity tests

$$\beta_k^\delta \in [0, \infty] \text{ max. values s.t. } \beta_k^\delta \Lambda'(1) \chi_{P_k} \geq \Lambda_{\text{meas}}^\delta - \delta I$$

- ▶ Monotonicity-constrained residuum minimization

$$\begin{aligned} & \|\Lambda'(1) \kappa^\delta - \Lambda_{\text{meas}}^\delta\|_F \rightarrow \min! \\ & \text{such that } \kappa^\delta|_{P_k} = \text{const.}, \quad 0 \leq \kappa^\delta|_{P_k} \leq \min\left\{\frac{1}{2}, \beta_k^\delta\right\} \end{aligned}$$

($\|\cdot\|_F$: Frobenius norm of Galerkin projektion to finite-dimensional space)

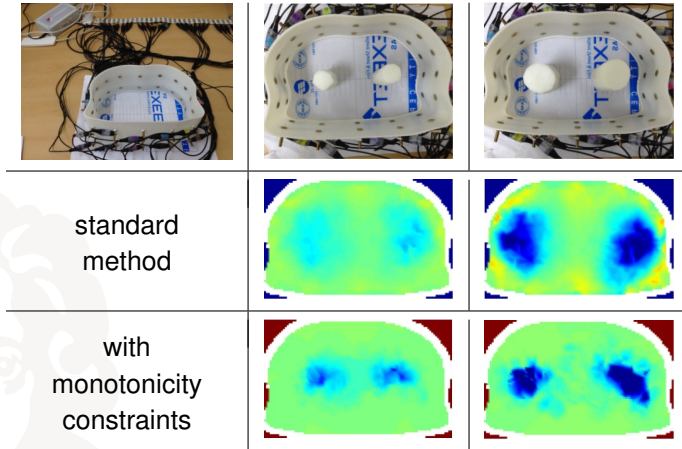
Theorem (H./Mach, submitted)

- ▶ For $\delta = 0$, there exists unique minimizer κ and

$$P_k \subseteq \text{supp } \kappa \iff P_k \subseteq \text{supp}(\sigma - 1).$$

- ▶ For noisy data, minimizers κ^δ exist and $\kappa^\delta \rightarrow \kappa$ pointwise.
-

Phantom experiment



Enhancing standard methods by monotonicity-based constraints

(Zhou/H./Seo, submitted)

Computational science and inverse problems

- ▶ Computational science is the core of many new advances.
- ▶ Inverse problems is the core of new medical imaging systems.

For ill-posed inverse problems

- ▶ Regularization is required for convergent algorithms.
- ▶ Regularization can also incorporate additional information (e.g., total variation penalization, stochastic priors, etc.)

For the non-linear ill-posed inverse problem of EIT

- ▶ Convergence of standard regul. techniques is still unclear.
- ▶ Monotonicity-based regularization allow fast, rigorous, and globally convergent reconstruction of shape information.