Inverse problems and medical imaging

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Introduction to inverse problems

Laplace's demon

Pierre Simon Laplace (1814):

"An intellect which ... would know
all forces ... and all positions of all items,

if this intellect were also vast enough to
submit these data to analysis ...

for such an intellect nothing would be
uncertain and the future just like the past
would be present before its eyes."

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Computational Science:

If we know all necessary parameters, then we can numerically predict the outcome of an experiment (by solving mathematical formulas).

Goals:

- Prediction
- Optimization
- Inversion/Identification

Generic simulation problem:

Given input x calculate outcome $y = F(x)$.

$x \in X$: parameters / input
$y \in Y$: outcome / measurements
$F: X \to Y$: functional relation / model

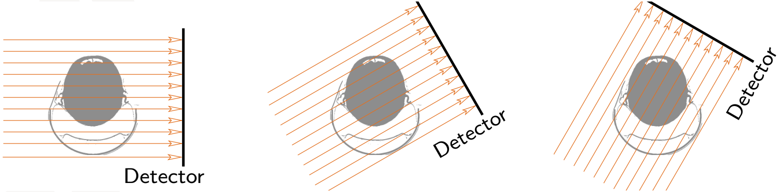
Goals:

- Prediction: Given x, calculate $y = F(x)$.
- Optimization: Find x, such that $F(x)$ is optimal.
- Inversion/Identification: Given $F(x)$, calculate x.

Example: X-ray computerized tomography (CT)

Nobel Prize in Physiology or Medicine 1979:
Allan M. Cormack and Godfrey N. Hounsfield

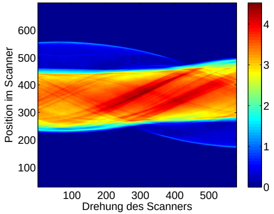
Idea: Take x-ray images from several directions



Computerized tomography (CT)

(Image: Hanke-Bourgeois, Grundlagen der Numerischen Mathematik und des Wiss. Rechnens, Teubner 2002)

$$\overset{F}{\longmapsto}$$



Image

Measurements

Direct problem: Simulate/predict the measurements

(from knowledge of the interior density distribution)

Given x *calculate* $F(x) = y$*!*

Inverse problem: Reconstruct/image the interior distribution

(from taking x-ray measurements)

Given y *solve* $F(x) = y$*!*

Computerized tomography

- CT forward operator $F: x \mapsto y$ is linear

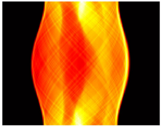
$\rightsquigarrow$ Evaluation of F is simple matrix vector multiplication

(after discretizing image and measurements as long vectors)

Simple low resolution example:

$\overset{F}{\mapsto}$

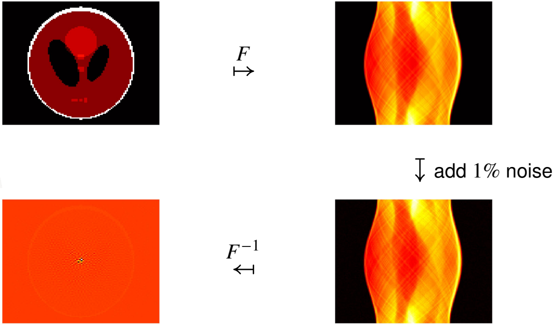
$\overset{F^{-1}}{\mapsfrom}$



Problem: Matrix F invertible, but $\|F^{-1}\|$ very large.

Ill-posedness

- In the continuous case: F^{-1} not continuous
- After discretization: $\|F^{-1}\|$ very large



Are stable reconstructions impossible?

Ill-posedness

Generic linear ill-posed inverse problem

- $F: X \to Y$ bounded and linear, X, Y Hilbert spaces,
- F injective, F^{-1} not continuous,
- True solution and noise-free measurements: $F\hat{x} = \hat{y}$,
- Real measurements: y^δ with $\|y^\delta - \hat{y}\| \leq \delta$

$$F^{-1}y^\delta \not\to F^{-1}\hat{y} = \hat{x} \quad \text{for} \quad \delta \to 0.$$

Even the smallest noise may corrupt the reconstructions.

Regularization

Generic linear Tikhonov regularization

$$R_\alpha = (F^*F + \alpha I)^{-1} F^*$$

$\rightsquigarrow R_\alpha$ continuous, $x = R_\alpha y^\delta$ minimizes

$$\|Fx - y^\delta\|^2 + \alpha \|x\|^2 \to \text{min!}$$

Theorem. Choose $\alpha := \delta$. Then for $\delta \to 0$,

$$R_\delta y^\delta \to F^{-1}\hat{y}.$$

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Proof. Show that $\|R_\alpha\| \leq \frac{1}{\sqrt{\alpha}}$ and apply

$$\|R_\alpha y^\delta - F^{-1}\hat{y}\| \leq \underbrace{\|R_\alpha(y^\delta - \hat{y})\|}_{\leq \|R_\alpha\|\,\delta} + \underbrace{\|R_\alpha\hat{y} - F^{-1}y\|}_{\to 0 \text{ for } \alpha \to 0}.$$

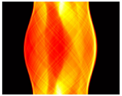
Inexact but continuous reconstruction (regularization)

\+ Information on measurement noise (parameter choice rule)

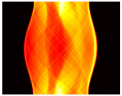
= Convergence

Example ($\delta = 1\%$)

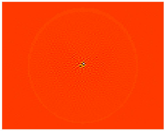
$\hat{x}$



$\hat{y} = F\hat{x}$



y^δ

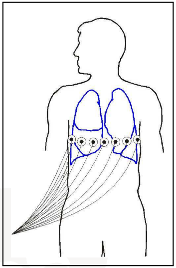
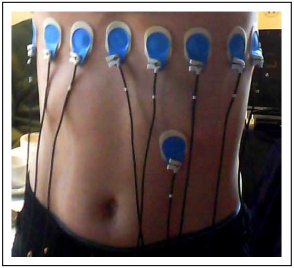
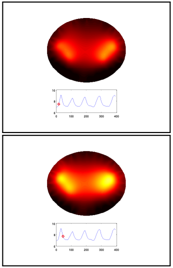


$F^{-1}y^\delta$

$(F^*F + \delta I)^{-1}F^*y^\delta$

Electrical impedance tomography

Electrical impedance tomography (EIT)

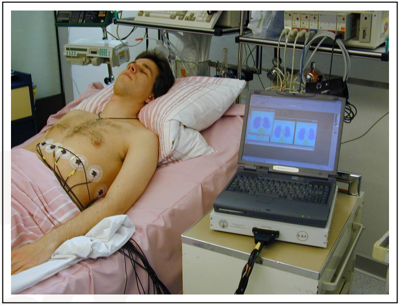
- Apply electric currents on subject's boundary
- Measure necessary voltages

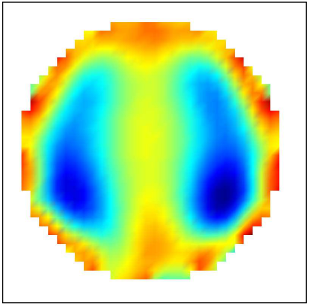
⇝ Reconstruct conductivity inside subject.

Images from BMBF-project on EIT

(Hanke, Kirsch, Kress, Hahn, Weller, Schilcher, 2007-2010)

MF-System Goe-MF II





Electric current strength: $5 - 500\mathrm{mA}_{\mathrm{rms}}$, 44 images/second,
CE certified by Viasys Healthcare, approved for clinical research

Mathematical Model

- Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega \qquad \text{(EIT)}$$

$\Omega \subset \mathbb{R}^n$:	imaged body, $n \geq 2$
$\sigma(x)$:	conductivity
$u(x)$:	electrical potential

- Idealistic model for boundary meas. (continuum model):

$\sigma \partial_\nu u(x)\|_{\partial\Omega}$:	applied electric current
$u(x)\|_{\partial\Omega}$:	measured boundary voltage (potential)

- Neumann-to-Dirichlet-Operator:

$$\Lambda(\sigma): L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (EIT) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Electrical impedance tomography

Inverse problem of EIT: Recover σ from $\Lambda(\sigma)$

Challenges:

- Uniqueness
  - Is σ uniquely determined from ”perfect data” $\Lambda(\sigma)$?
- Non-linearity and ill-posedness
  - Reconstruction algorithms to determine σ from $\Lambda(\sigma)$?
  - Local/global convergence results for noisy data $\Lambda^{\delta}_{\text{meas}} \approx \Lambda(\sigma)$?
- Realistic data
  - What can we recover from real measurements?

    (fixed number of electrodes, realistic electrode models, . . .)
  - Measurement and modelling errors? Resolution?

Inversion of $\sigma \mapsto \Lambda(\sigma) = \Lambda_{\text{meas}}$?

Generic solvers for non-linear inverse problems:

- Linearize and regularize:

$$\Lambda_{\text{meas}} = \Lambda(\sigma) \approx \Lambda(\sigma_0) + \Lambda'(\sigma_0)(\sigma - \sigma_0).$$

  σ_0: Initial guess or reference state (e.g. exhaled state)

  $\rightsquigarrow$ Linear inverse problem for σ

  (Solve, e.g., using linear Tikhonov regul., repeat for Newton-type algorithm.)

- Regularize and linearize:
  E.g., minimize non-linear Tikhonov functional

$$\|\Lambda_{\text{meas}} - \Lambda(\sigma)\|^2 + \alpha\|\sigma - \sigma_0\|^2 \to \text{min!}$$

Very flexible, but high comput. cost and convergence unclear

Linearization and shape reconstruction

Theorem (**H.**/Seo, SIAM J. Math. Anal. 2010)
Let κ, σ, σ_0 pcw. analytic.

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \quad \Longrightarrow \quad \operatorname{supp}_{\partial\Omega}\kappa = \operatorname{supp}_{\partial\Omega}(\sigma - \sigma_0)$$

$\operatorname{supp}_{\partial\Omega}$: outer support ($=\operatorname{supp}$ + parts unreachable from $\partial\Omega$)

- Linearized EIT equation contains correct shape information
- For the shape reconstruction problem $\Lambda(\sigma) \mapsto \operatorname{supp}_{\partial\Omega}(\sigma - \sigma_0)$ fast, rigorous and globally convergent method seem possible.
- Goal: Given $\Lambda^\delta_{\text{meas}} \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, can we regularize

$$\|\Lambda'(\sigma_0)\kappa - \Lambda^\delta_{\text{meas}}\| \to \min!$$

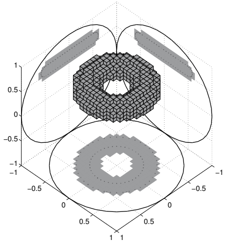
so that $\operatorname{supp}_{\partial\Omega}\kappa^\delta \to \operatorname{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

Monotonicity method (for simple test example)

Theorem *(**H.**/Ullrich, SIAM J. Math. Anal. 2013)*

$\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1+\chi_B) \geq \Lambda(\sigma).$$



For faster implementation:

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1) + \tfrac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$

Shape can be reconstructed by linearized monotonicity tests.

$\rightsquigarrow$ fast, rigorous, allows globally convergent implementation

Sketch of proof

Theorem $\Omega \setminus \overline{D}$ connected, B open.

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1+\chi_B) \geq \Lambda(1+\chi_D).$$

„$\Longrightarrow$": follows from monotonicity inequality:

$$\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \geq \int_{\partial\Omega} g\,(\Lambda(\sigma_0) - \Lambda(\sigma_1))\,g \geq \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)|\nabla u_0|^2$$

„$\Longleftarrow$": follows from using localized potentials in monoton. inequality.
If $B \not\subseteq D$ then there exist solutions $u_0^{(k)}$, $k \in \mathbb{N}$ with

$$\int_B \left|\nabla u_0^{(k)}\right|^2 \mathrm{d}x \to \infty \quad \text{and} \quad \int_D \left|\nabla u_0^{(k)}\right|^2 \mathrm{d}x \to 0.$$

Improving residuum-based methods

Let $\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

- Pixel partition $\Omega = \bigcup_{k=1}^{m} P_k$
- Regularized monotonicity tests

$$\beta_k^\delta \in [0, \infty] \text{ max. values s.t. } \beta_k^\delta \Lambda'(1)\chi_{P_k} \geq \Lambda_{\text{meas}}^\delta - \delta I$$

- Monotonicity-constrained residuum minimization

$$\|\Lambda'(1)\kappa^\delta - \Lambda_{\text{meas}}^\delta\|_{\text{F}} \to \text{min!}$$

$$\text{such that} \quad \kappa^\delta|_{P_k} = \text{const.},\ 0 \leq \kappa^\delta|_{P_k} \leq \min\{\tfrac{1}{2}, \beta_k^\delta\}$$

($\|\cdot\|_F$: Frobenius norm of Galerkin projektion to finite-dimensional space)

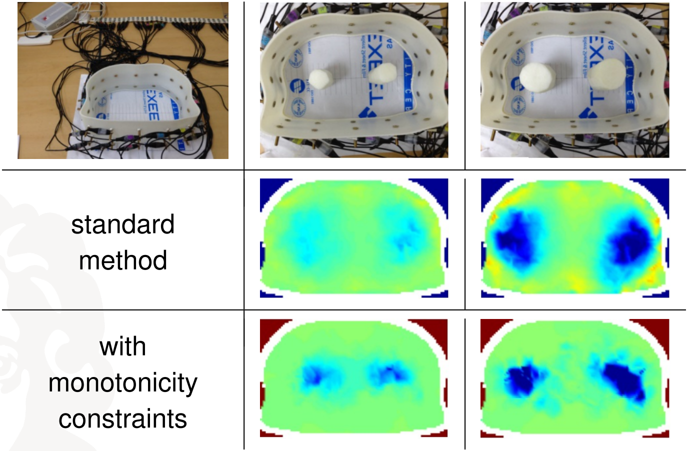
Theorem (**H.**/Minh, Inverse Problems 2016)

- For $\delta = 0$, there exists unique minimizer κ and

$$P_k \subseteq \operatorname{supp} \kappa \quad \Longleftrightarrow \quad P_k \subseteq \operatorname{supp}(\sigma - 1).$$

- For noisy data, minimizers κ^δ exist and $\kappa^\delta \to \kappa$ pointwise.

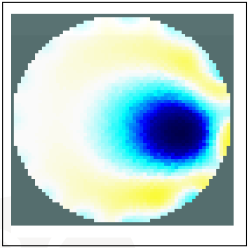
Phantom experiment



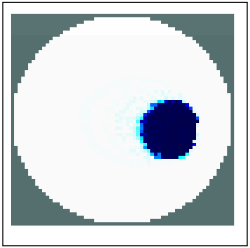
Enhancing standard methods by (heuristic) monotonicity constraints

(Zhou/**H.**/Seo)

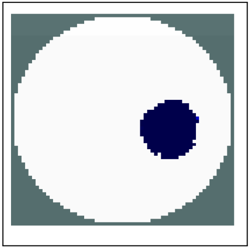
Benchmark example



standard



monoton.-constrained
(Matlab quadprog)



monoton.-constrained
(cvx package)

Rigorous monoton.-constrained method vs. community standard

(**H.**/Minh)

- EIT community standard: `inv_solve` in EIDORS
- EIDORS: `http://eidors3d.sourceforge.net` (Adler/Lionheart)
- Dataset: `iirc_data_2006` (Woo et al.): 2cm insulated inclusion in 20cm tank
  - using interpolated data on active electrodes (**H.**, Inverse Problems 2015)

Conclusions

Computational science and inverse problems

- Computational science is the core of many new advances.
- Inverse problems is the core of new medical imaging systems.

For ill-posed inverse problems

- Regularization is required for convergent algorithms.
- Regularization can also incorporate additional information

  (e.g., total variation penalization, stochastic priors, etc.)

For the non-linear ill-posed inverse problem of EIT

- Convergence of standard regularization is still unclear.
- Monotonicity-based regularization allows fast, rigorous, and globally convergent reconstruction of shape information.