Introduction to Inverse Problems

Bastian von Harrach

harrach@math.uni-stuttgart.de

Chair of Optimization and Inverse Problems, University of Stuttgart, Germany

Department of Computational Science & Engineering
Yonsei University
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Motivation and examples

Laplace's demon

Laplace's demon: *(Pierre Simon Laplace 1814)*

"An intellect which (. . .) would know all forces (. . .) and all positions of all items (. . .), if this intellect were also vast enough to submit these data to analysis, (. . .); for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes."

Computational Science

Computational Science / Simulation Technology:

If we know all necessary parameters, then we can numerically predict the outcome of an experiment (by solving mathematical formulas).

Goals:

- Prediction
- Optimization
- Inversion/Identification

Computational Science

Generic simulation problem:

Given input x calculate outcome $y = F(x)$.

$$
\begin{aligned}
x \in X: &\quad \text{parameters / input}\\
y \in Y: &\quad \text{outcome / measurements}\\
F: X \to Y: &\quad \text{functional relation / model}
\end{aligned}
$$

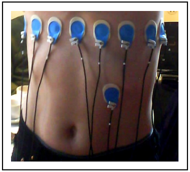
Goals:

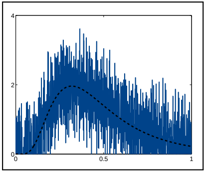
- Prediction: Given x, calculate $y = F(x)$.
- Optimization: Find x, such that $F(x)$ is optimal.
- Inversion/Identification: Given $F(x)$, calculate x.

Examples

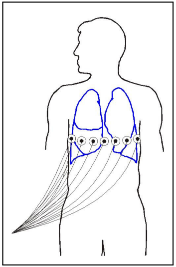
Examples of inverse problems:

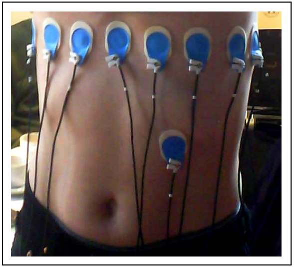
- Electrical impedance tomography
- Computerized tomography
- Image Deblurring
- Numerical Differentiation

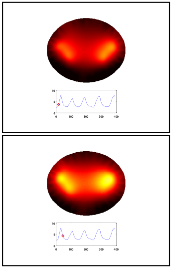




Electrical impedance tomography (EIT)







- Apply electric currents on subject's boundary
- Measure necessary voltages

⇝ Reconstruct conductivity inside subject.

Electrical impedance tomography (EIT)

x Image	$\overset{F}{\longmapsto}$	$y = F(x)$ Measurements

x: Interior conductivity distribution (image)
y: Voltage and current measurements

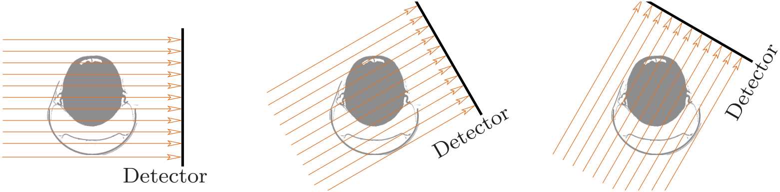
Direct problem: Simulate/predict the measurements
(from knowledge of the interior conductivity distribution)
Given x calculate $F(x) = y$!

Inverse problem: Reconstruct/image the interior distribution
(from taking voltage/current measurements)
Given y solve $F(x) = y$!

X-ray computerized tomography

Nobel Prize in Physiology or Medicine 1979:
Allan M. Cormack and Godfrey N. Hounsfield
(Photos: Copyright ©The Nobel Foundation)

Idea: Take x-ray images from several directions



Computerized tomography (CT)

$\overset{F}{\longmapsto}$

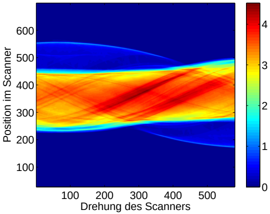


Image | Measurements

Direct problem: Simulate/predict the measurements

(from knowledge of the interior density distribution)

Given x *calculate* $F(x) = y$*!*

Inverse problem: Reconstruct/image the interior distribution

(from taking x-ray measurements)

Given y *solve* $F(x) = y$*!*

Image deblurring

x

True image

$\overset{F}{\longmapsto}$

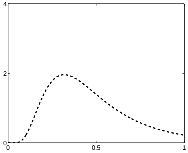
$y = F(x)$

Blurred image

Direct problem: Simulate/predict the blurred image
(from knowledge of the true image)
Given x calculate $F(x) = y$!

Inverse problem: Reconstruct/image the true image
(from the blurred image)
Given y solve $F(x) = y$!

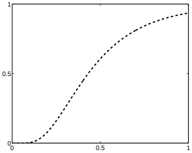
Numerical differentiation



x

Function

$\overset{F}{\longmapsto}$



$y = F(x)$

Primitive Function

Direct problem: Calculate the primitive
Given x calculate $F(x) = y$!

Inverse problem: Calculate the derivative
Given y solve $F(x) = y$!

Ill-posedness

Well-posedness

Hadamard (1865–1963): A problem is called well-posed, if

- a solution exists,
- the solution is unique,
- the solution depends continuously on the given data.

Inverse Problem: *Given y solve $F(x) = y$!*

- F surjective?
- F injective?
- F^{-1} continuous?

Ill-posed problems

Ill-posedness: $F^{-1}: \ Y \to X$ not continuous.

$\hat{x} \in X$: true solution

$\hat{y} = F(\hat{x}) \in Y$: exact measurement

$y^\delta \in Y$: real measurement containing noise $\delta > 0$,

e.g. $\|y^\delta - \hat{y}\|_Y \leq \delta$

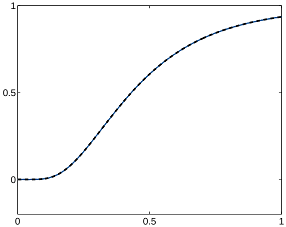
For $\delta \to 0$

$$y^\delta \to \hat{y}, \quad \text{but (generally)} \quad F^{-1}(y^\delta) \not\to F^{-1}(\hat{y}) = \hat{x}$$

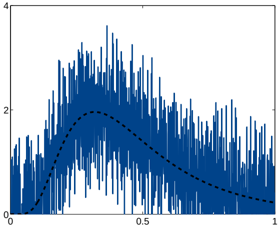
Even the smallest amount of noise will corrupt the reconstructions.

Numerical differentiation

Numerical differentiation example ($h = 10^{-3}$)



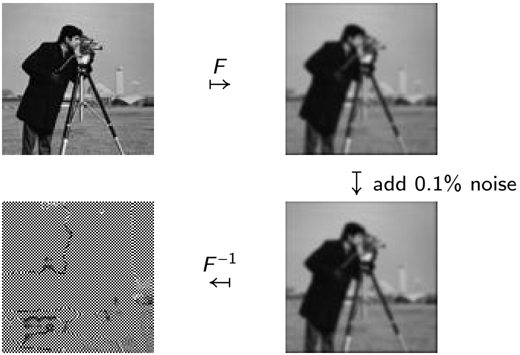
$y(t)$ and $y^\delta(t)$



$\frac{y(t+h)-y(t)}{h}$ and $\frac{y^\delta(t+h)-y^\delta(t)}{h}$

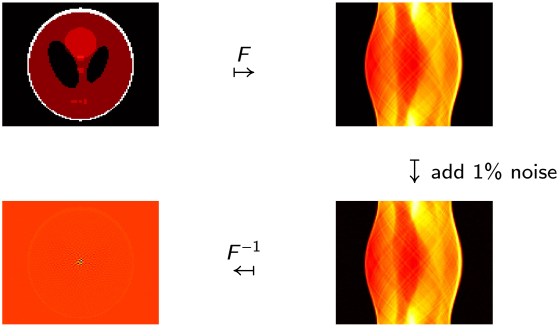
Differentiation seems to be an ill-posed (inverse) problem.

Image deblurring



Deblurring seems to be an ill-posed (inverse) problem.

Image deblurring



CT seems to be an ill-posed (inverse) problem.

Compactness and ill-posedness

Compactness

Consider the general problem

$$F: \ X \to Y, \quad F(x) = y$$

with X, Y real Hilbert spaces.
Assume that F is linear, bounded and injective with left inverse

$$F^{-1}: \ F(X) \subseteq Y \to X.$$

Definition 1.1. $F \in \mathcal{L}(X, Y)$ is called compact, if

$$\overline{F(U)} \text{ is compact for alle bounded } U \subseteq X,$$

i.e. if $(x_n)_{n\in\mathbb{N}} \subset X$ is a bounded sequence then $(F(x_n))_{n\in\mathbb{N}} \subset Y$ contains a bounded subsequence.

Compactness

Theorem 1.2. Let

- $F \in \mathcal{L}(X, Y)$ be compact and injective, and
- $\dim X = \infty$,

then the left inverse F^{-1} is not continuous, i.e. the inverse problem

$$Fx = y$$

is ill-posed.

Compactness

Theorem 1.3. Every limit[1] of compact operators is compact.

Theorem 1.4. If $\dim \mathcal{R}(F) < \infty$ then F is compact.

Corollary. Every operator that can be approximated[1] by finite dimensional operators is compact.

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[1] in the uniform operator topology

Compactness

Theorem 1.5. Let $F \in \mathcal{L}(X, Y)$ possess an unbounded left inverse F^{-1}, and let $R_n \in \mathcal{L}(Y, X)$ be a sequence with

$$R_n y \to F^{-1} y \quad \text{for all } y \in \mathcal{R}(F).$$

Then $\|R_n\| \to \infty$.

Corollary. If we discretize an ill-posed problem, the better we discretize, the more unbounded our discretizations become.

Compactness and ill-posedness

Discretization: Approximation by finite-dimensional operators.

Consequences for discretizing infinite-dimensional problems:

If an infinite-dimensional direct problem can be discretized[1], then

- the direct operator is compact.
- the inverse problem is ill-posed, i.e. the smallest amount of measurement noise may completely corrupt the outcome of the (exact, infinite-dimensional) inversion.

If we discretize the inverse problem, then

- the better we discretize, the larger the noise amplification is.

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[1] in the uniform operator topology

Examples

- The operator

$$F: \quad \text{function} \mapsto \text{primitive function}$$

is a linear, compact operator.

$\rightsquigarrow$ The inverse problem of differentiation is ill-posed.

- The operator

$$F: \quad \text{exact image} \mapsto \text{blurred image}$$

is a linear, compact operator.

$\rightsquigarrow$ The inverse problem of image deblurring is ill-posed.

Examples

- In computerized tomography, the operator

  $$F: \quad \text{image} \ \mapsto \ \text{measurements}$$

  is a linear, compact operator.

  $\rightsquigarrow$ The inverse problem of CT is ill-posed.

- In EIT, the operator

  $$F: \quad \text{image} \ \mapsto \ \text{measurements}$$

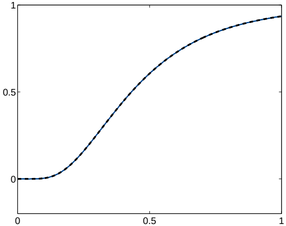
  is a non-linear operator. Its Fréchet derivative is a compact linear operator.

  $\rightsquigarrow$ The (linearized) inverse problem of EIT is ill-posed.

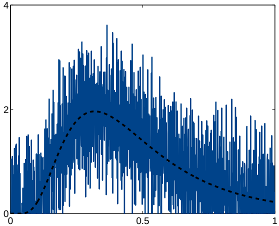
Regularization

Numerical differentiation

Numerical differentiation example



$y(t)$ and $y^{\delta}(t)$



$\frac{y(t+h)-y(t)}{h}$ and $\frac{y^{\delta}(t+h)-y^{\delta}(t)}{h}$

Differentiation is an ill-posed (inverse) problem

Regularization

Numerical differentiation:

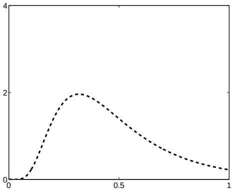
- $y \in C^2$, $C := 2\sup_\tau |g''(\tau)| < \infty$, $|y^\delta(t) - y(t)| \leq \delta \ \forall t$

$$\left| y'(t) - \frac{y^\delta(t+h) - y^\delta(t)}{h} \right|$$
$$\leq \left| y'(x) - \frac{y(t+h) - y(t)}{h} \right|$$
$$+ \left| \frac{y(t+h) - y(t)}{h} - \frac{y^\delta(t+h) - y^\delta(t)}{h} \right|$$
$$\leq Ch + \frac{2\delta}{h} \to 0.$$

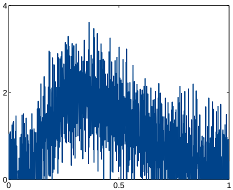
for $\delta \to 0$ and adequately chosen $h = h(\delta)$, e.g., $h := \sqrt{\delta}$.

Numerical differentiation

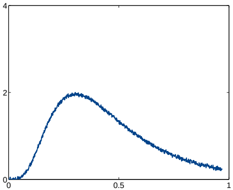
Numerical differentiation example



$y'(t)$



$\frac{y^\delta(t+h)-y^\delta(t)}{h}$
with h very small



$\frac{y^\delta(t+h)-y^\delta(t)}{h}$
with $h \approx \sqrt{\delta}$

Idea of regularization: Balance noise amplification and approximation

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Regularization

Regularization of inverse problems:

- F^{-1} not continuous, so that generally $F^{-1}(y^\delta) \not\to F^{-1}(y) = x$ for $\delta \to 0$
- R_α continuous approximations of F^{-1}

  $R_\alpha \to F^{-1}$ (pointwise) for $\alpha \to 0$

$$R_{\alpha(\delta)} y^\delta \to F^{-1} y = x \quad \text{for } \delta \to 0$$

if the parameter $\alpha = \alpha(\delta)$ is correctly chosen.

Inexact but continuous reconstruction (regularization)
+ Information on measurement noise (parameter choice rule)
= Convergence

Prominent regularization methods

- Tikhonov regularization

$$R_\alpha = (F^* F + \alpha I)^{-1} F^*$$

  - R_α continuous, $\|R_\alpha\| \leq \frac{1}{\sqrt{\alpha}}$
  - $R_\alpha y \to F^{-1} y$ for $\alpha \to 0$, $y \in \mathcal{R}(F)$
  - $R_\alpha y^\delta$ minimizes

$$\left\| Fx - y^\delta \right\|^2 + \alpha \left\| x \right\|^2 \to \text{min!}$$

- Truncated singular value decomposition (TSVD)
- Landweber method
- ...

Parameter choice rule

Convergence of Tikhonov-regularization

$$\left\| R_\alpha y^\delta - F^{-1} y \right\| \leq \underbrace{\left\| R_\alpha (y^\delta - y) \right\|}_{\leq \|R_\alpha\| \delta} + \underbrace{\left\| R_\alpha y - F^{-1} y \right\|}_{\to 0 \text{ for } \alpha \to 0}$$

Choose $\alpha(\delta)$ such that (for $\delta \to 0$)

- $\alpha(\delta) \to 0$
- $\left\| R_{\alpha(\delta)} \right\| \delta = \frac{\delta}{\sqrt{\alpha(\delta)}} \to 0$

then $R_{\alpha(\delta)} y^\delta \to F^{-1} y$. E.g., set $\alpha(\delta) := \delta$.

Exakt inversion does not converge, $F^{-1} y^\delta \not\to F^{-1} y$.
Tikhonov-regularization converges, $R_\delta y^\delta \to F^{-1} y$.

Better parameter choice rule:

Choose α such that $\left\| F R_\alpha y^\delta - y^\delta \right\| \approx \delta$ (discrepancy principle)

Image deblurring

$\hat{x}$

$\hat{y} = F\hat{x}$

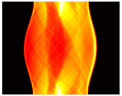
y^δ

$F^{-1}y^\delta$

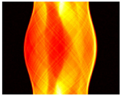
$(F^*F + \delta I)^{-1}F^*y^\delta$

Computerized tomography

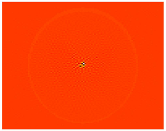
$\hat{x}$



$\hat{y} = F\hat{x}$



y^δ



$F^{-1} y^\delta$

$(F^*F + \delta I)^{-1} F^* y^\delta$

Conclusions and remarks

Conclusions

- Inverse problems are of great importance in comput. science.
  (parameter identification, medical tomography, etc.)
- For ill-posed inverse problems, the best data-fit solutions generally do not converge against the true solution.
- The regularized solutions do converge against the true solution.

Strategies for non-linear inverse problems $F(x) = y$:

- First linearize, then regularize.
- First regularize, then linearize.

A-priori information

- Regularization can be used to incorporate a-priori knowledge (promote sparsity or sharp edges, include stochastic priors, etc.)