

Inverse problems and medical imaging

Bastian von Harrach

`harrach@math.uni-frankfurt.de`

Institute of Mathematics, Goethe University Frankfurt, Germany

Colloquium of the Department of Mathematics
Saarland University, Saarbrücken, Germany

November 27, 2015

Introduction to inverse problems

Pierre Simon Laplace (1814):

*"An intellect which ... would know
all forces ... and all positions of all items,
if this intellect were also vast enough to
submit these data to analysis ...*

*for such an intellect nothing would be
uncertain and the future just like the past
would be present before its eyes."*



Computational Science / Simulation Technology:

If we know all necessary parameters, then we can numerically predict the outcome of an experiment (by solving mathematical formulas).

Goals:

- ▶ Prediction
- ▶ Optimization
- ▶ Inversion/Identification

Generic simulation problem:

Given input x calculate outcome $y = F(x)$.

$x \in X$: parameters / input

$y \in Y$: outcome / measurements

$F : X \rightarrow Y$: functional relation / model

Goals:

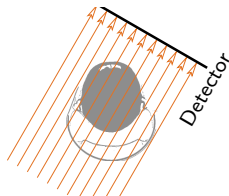
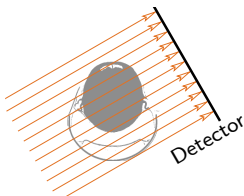
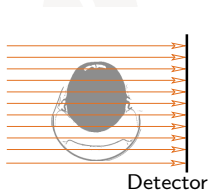
- ▶ **Prediction:** Given x , calculate $y = F(x)$.
- ▶ **Optimization:** Find x , such that $F(x)$ is optimal.
- ▶ **Inversion/Identification:** Given $F(x)$, calculate x .

Example: X-ray computerized tomography (CT)

Nobel Prize in Physiology or Medicine 1979:
Allan M. Cormack and Godfrey N. Hounsfield
(Photos: Copyright ©The Nobel Foundation)



Idea: Take x-ray images from several directions



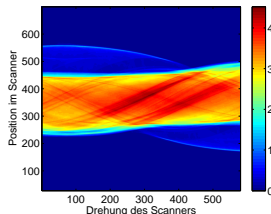
Computerized tomography (CT)

(Image: Hanke-Bourgeois, Grundlagen der Numerischen Mathematik und des Wiss. Rechnens, Teubner 2002)



Image

F



Measurements

Direct problem:

Simulate/predict the measurements

(from knowledge of the interior density distribution)

Given x calculate $F(x) = y!$

Inverse problem:

Reconstruct/image the interior distribution

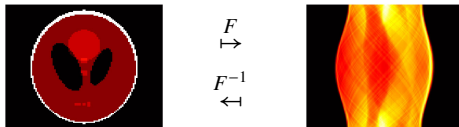
(from taking x-ray measurements)

Given y solve $F(x) = y!$

Computerized tomography

- ▶ CT forward operator $F : x \mapsto y$ is linear
- ↷ Evaluation of F is simple matrix vector multiplication
(after discretizing image and measurements as long vectors)

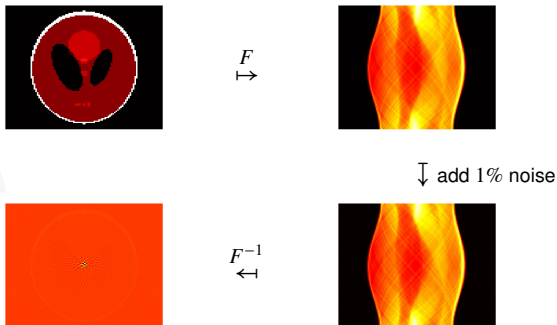
Simple low resolution example:



Problem: Matrix F invertible, but $\|F^{-1}\|$ very large.

Ill-posedness

- ▶ In the continuous case: F^{-1} not continuous
- ▶ After discretization: $\|F^{-1}\|$ very large



Are stable reconstructions impossible?

Ill-posedness

Generic linear ill-posed inverse problem

- ▶ $F : X \rightarrow Y$ bounded and linear, X, Y Hilbert spaces,
- ▶ F injective, F^{-1} not continuous,
- ▶ True solution and noise-free measurements: $F\hat{x} = \hat{y}$,
- ▶ Real measurements: y^δ with $\|y^\delta - \hat{y}\| \leq \delta$

$$F^{-1}y^\delta \not\rightarrow F^{-1}\hat{y} = \hat{x} \quad \text{for} \quad \delta \rightarrow 0.$$

Even the smallest noise may corrupt the reconstructions.

Generic linear Tikhonov regularization

$$R_{\alpha} = (F^*F + \alpha I)^{-1}F^*$$

$\leadsto R_{\alpha}$ continuous, $R_{\alpha}y^{\delta}$ minimizes

$$\|Fx - y^{\delta}\|^2 + \alpha \|x\|^2 \rightarrow \min!$$

Theorem. Choose $\alpha := \delta$. Then for $\delta \rightarrow 0$,

$$R_{\delta}y^{\delta} \rightarrow F^{-1}\hat{y}.$$

Regularization

Theorem. Choose $\alpha := \delta$. Then for $\delta \rightarrow 0$,

$$R_\delta y^\delta \rightarrow F^{-1} \hat{y}.$$

Proof. Show that $\|R_\alpha\| \leq \frac{1}{\sqrt{\alpha}}$ and apply

$$\|R_\alpha y^\delta - F^{-1} \hat{y}\| \leq \underbrace{\|R_\alpha(y^\delta - y)\|}_{\leq \|R_\alpha\| \delta} + \underbrace{\|R_\alpha y - F^{-1} y\|}_{\rightarrow 0 \text{ for } \alpha \rightarrow 0}.$$

Inexact but continuous reconstruction (**regularization**)

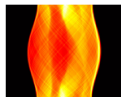
+ Information on measurement noise (**parameter choice rule**)

= Convergence

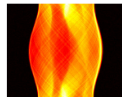
Example



\hat{x}



$\hat{y} = F \hat{x}$



y^δ



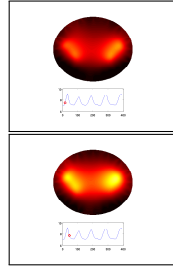
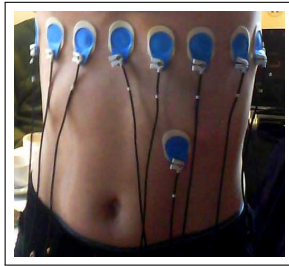
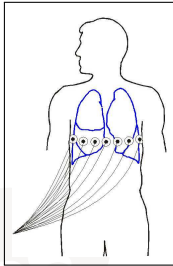
$F^{-1} y^\delta$



$(F^* F + \delta I)^{-1} F^* y^\delta$

Electrical impedance tomography

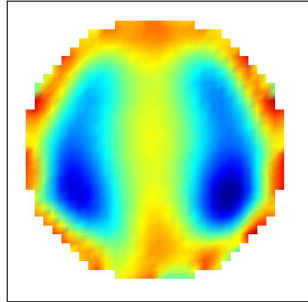
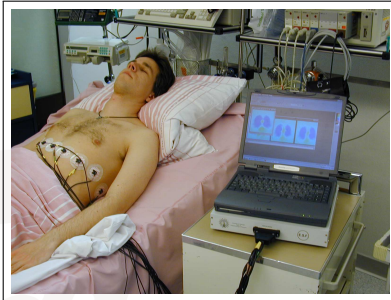
Electrical impedance tomography (EIT)



- ▶ Apply electric currents on subject's boundary
- ▶ Measure necessary voltages
- ~> Reconstruct conductivity inside subject.

Images from BMBF-project on EIT

(Hanke, Kirsch, Kress, Hahn, Weller, Schilcher, 2007-2010)



Electric current strength: $5 - 500 \text{ mA}_{\text{rms}}$, 44 images/second,
CE certified by Viasys Healthcare, approved for clinical research

- ▶ Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega \quad (\text{EIT})$$

$\Omega \subset \mathbb{R}^n$: imaged body, $n \geq 2$

$\sigma(x)$: conductivity

$u(x)$: electrical potential

- ▶ Idealistic model for boundary meas. (continuum model):

$\sigma \partial_\nu u(x)|_{\partial\Omega}$: applied electric current

$u(x)|_{\partial\Omega}$: measured boundary voltage (potential)

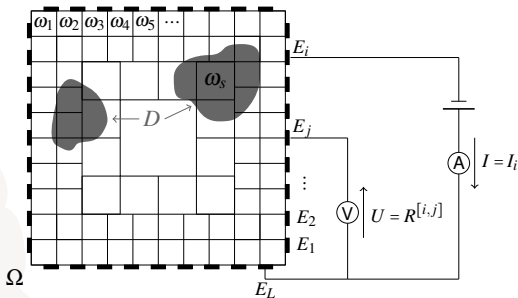
- ▶ Neumann-to-Dirichlet-Operator:

$$\Lambda(\sigma) : L^2_\diamond(\partial\Omega) \rightarrow L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (EIT) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Realistic electrode model

- Complete electrode model including contact impedances z_l
(for $z_l = 0$: shunt electrode model)



$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \Omega$$

$$u|_{E_j} + z_l \sigma \partial_{\nu} u|_{E_j} = \text{const.} =: R^{[i,j]}(\sigma)$$

$$\sigma \partial_{\nu} u|_{\partial \Omega} = 0 \quad \text{outside } E_l$$

$$\int_{E_l} \sigma \partial_{\nu} u \, ds = I(\delta_{i,l} - \delta_{j,l})$$

Electrical impedance tomography

Roughly speaking, $R(\sigma) \rightarrow \Lambda(\sigma)$ for $\# \text{electrodes} \rightarrow \infty$

Inverse problem of EIT: Recover σ from $\Lambda(\sigma)$, resp., $R(\sigma)$

Challenges:

- ▶ Uniqueness
 - ▶ Is σ uniquely determined from "perfect data" $\Lambda(\sigma)$?
- ▶ Non-linearity and ill-posedness
 - ▶ Reconstruction algorithms to determine σ from $\Lambda(\sigma)$?
 - ▶ Local/global convergence results when using noisy $R(\sigma)$?
- ▶ Realistic data
 - ▶ What can we recover from real measurements?
(fixed number of electrodes, realistic electrode models, ...)
 - ▶ Measurement and modelling errors? Resolution?

Inversion of $\sigma \mapsto \Lambda(\sigma) = \Lambda_{\text{meas}}$?

Generic solvers for non-linear inverse problems:

- ▶ **Linearize and regularize:**

$$\Lambda_{\text{meas}} = \Lambda(\sigma) \approx \Lambda(\sigma_0) + \Lambda'(\sigma_0)(\sigma - \sigma_0).$$

σ_0 : Initial guess or reference state (e.g. exhaled state)

↪ Linear inverse problem for σ

(Solve, e.g., using linear Tikhonov regul., repeat for Newton-type algorithm.)

- ▶ **Regularize and linearize:**

E.g., minimize non-linear Tikhonov functional

$$\|\Lambda_{\text{meas}} - \Lambda(\sigma)\|^2 + \alpha \|\sigma - \sigma_0\|^2 \rightarrow \min!$$

Very flexible, but high comput. cost and convergence unclear

Linearization and shape reconstruction

Theorem (H./Seo, SIAM J. Math. Anal. 2010)

Let κ , σ , σ_0 pcw. analytic.

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \implies \text{supp}_{\partial\Omega}\kappa = \text{supp}_{\partial\Omega}(\sigma - \sigma_0)$$

$\text{supp}_{\partial\Omega}$: outer support (= supp + parts unreachable from $\partial\Omega$)

- ▶ Linearized EIT equation contains correct shape information
- ▶ For the shape reconstruction problem

$$\Lambda(\sigma) \mapsto \text{supp}_{\partial\Omega}(\sigma - \sigma_0)$$

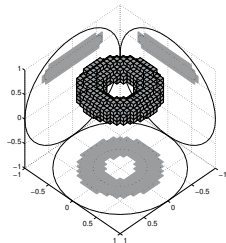
fast, rigorous and globally convergent method seem possible.

Monotonicity method (for simple test example)

Theorem (H./Ullrich, SIAM J. Math. Anal. 2013)

$\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

$$B \subseteq D \iff \Lambda(1 + \chi_B) \geq \Lambda(\sigma).$$



For faster implementation:

$$B \subseteq D \iff \Lambda(1) + \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$

Shape can be reconstructed by linearized monotonicity tests.

~> fast, rigorous, allows globally convergent implementation

For real electrode measurements $R(\sigma)$:

" \implies ": still holds

" \impliedby ": holds if "enough" electrodes are used

Improving residuum-based methods

Theorem (H./Minh, submitted)

Let $\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

- ▶ Pixel partition $\Omega = \bigcup_{k=1}^m P_k$
- ▶ Monotonicity tests

$\beta_k \in [0, \infty]$ max. values s.t. $\beta_k \Lambda'(1) \chi_{P_k} \geq \Lambda(\sigma) - \Lambda(1)$

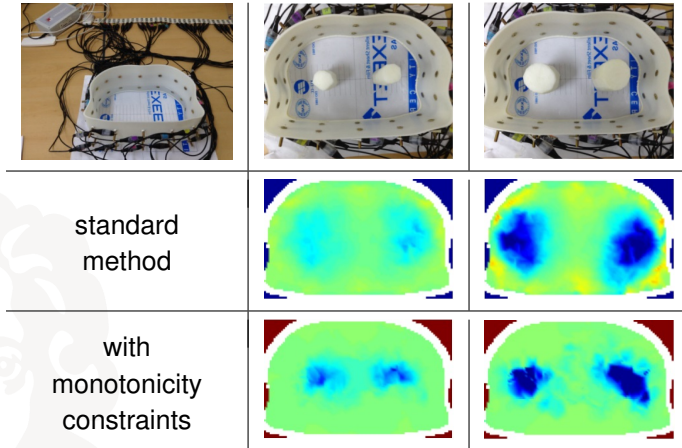
- ▶ $L(\kappa) \in \mathbb{R}^{s \times s}$: Discretization of lin. residual $\Lambda(\sigma) - \Lambda(1) - \Lambda'(1)\kappa$
(e.g. Galerkin proj. to fin.-dim. space)

Then, the monotonicity-constrained residuum minimization problem

$$\|L(\kappa)\|_F \rightarrow \min! \quad \text{s.t.} \quad \kappa|_{P_k} = \text{const.}, \quad 0 \leq \kappa|_{P_k} \leq \min\{\frac{1}{2}, \beta_k\}$$

possesses a unique solution κ , and $P_k \subseteq \text{supp } \kappa$ iff $P_k \subseteq \text{supp}(\sigma - 1)$.

Phantom experiment



Enhancing standard methods by monotonicity-based constraints

(Zhou/H./Seo, submitted)

Computational science and inverse problems

- ▶ Computational science is the core of many new advances.
- ▶ Inverse problems is the core of new medical imaging systems.

For ill-posed inverse problems

- ▶ Regularization is required for convergent algorithms.
- ▶ Regularization can also incorporate additional information (e.g., total variation penalization, stochastic priors, etc.)

For the non-linear ill-posed inverse problem of EIT

- ▶ Convergence of standard regul. techniques is still unclear.
- ▶ Monotonicity-based regularization allow fast, rigorous, and globally convergent reconstruction of shape information.