Monotonicity-based methods for elliptic inverse coefficient problems

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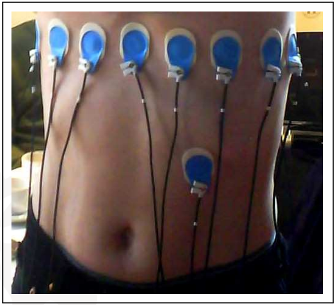
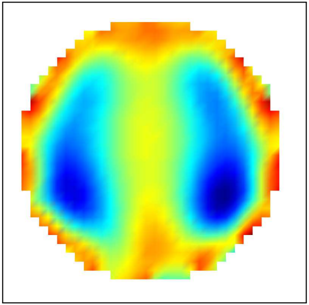
Electrical impedance tomography (EIT)





- Apply electric currents on subject's boundary
- Measure necessary voltages

$\rightsquigarrow$ Reconstruct conductivity inside subject.

Mathematical Model

Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega$$

$\Omega \subset \mathbb{R}^n$: imaged body, $n \geq 2$
$\sigma(x)$: conductivity
$u(x)$: electrical potential

Idealistic model for boundary measurements (continuum model):

$\sigma \partial_\nu u(x)|_{\partial\Omega}$: applied electric current
$u(x)|_{\partial\Omega}$: measured boundary voltage (potential)

Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves (1)}\} \text{ ?}$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma) : \ L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Inversion of $\sigma \mapsto \Lambda(\sigma) = \Lambda_{\text{meas}}$?

Generic solvers for non-linear inverse problems:

- Linearize and regularize:

  $$\Lambda_{\text{meas}} = \Lambda(\sigma) \approx \Lambda(\sigma_0) + \Lambda'(\sigma_0)(\sigma - \sigma_0).$$

  σ_0: Initial guess or reference state (e.g. exhaled state)

  $\rightsquigarrow$ Linear inverse problem for σ

  (Solve using linear regularization method, repeat for Newton-type algorithm.)

- Regularize and linearize:

  E.g., minimize non-linear Tikhonov functional

  $$\left\| \Lambda_{\text{meas}} - \Lambda(\sigma) \right\|^2 + \alpha \left\| \sigma - \sigma_0 \right\|^2 \to \text{min!}$$

Advantages of generic solvers:

- Very flexible, additional data/unknowns easily incorporated
- Problem-specific regularization can be applied

  (e.g., total variation penalization, stochastic priors, etc.)

Inversion of $\sigma \mapsto \Lambda(\sigma) = \Lambda_{\text{meas}}$?

Problems with generic solvers

- High computational cost

  (Evaluations of $\Lambda(\cdot)$ and $\Lambda'(\cdot)$ require PDE solutions)

- Convergence unclear

  (Validity of TCC/Scherzer-condition is a long-standing open problem for EIT.)

  - Convergence against true solution for exact meas. Λ_{meas}?

    (in the limit of infinite computation time)

  - Convergence against true solution for noisy meas. $\Lambda_{\text{meas}}^{\delta}$?

    (in the limit of $\delta \to 0$ and infinite computation time)

  - Global convergence? Resolution estimates for realistic noise?

D-bar method

- Convergent 2D-implementation for $\sigma \in C^2$ and full bndry data

  (Knudsen, Lassas, Mueller, Siltanen 2008)

Linearization and shape reconstruction

Theorem (**H.**/Seo, SIMA 2010)
Let κ, σ, σ_0 pcw. analytic.

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \quad \Longrightarrow \quad \mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$$

$\mathrm{supp}_{\partial\Omega}$: outer support ($=$ supp $+$ parts unreachable from $\partial\Omega$)

- $\rightsquigarrow$ Linearized EIT equation contains correct shape information

Next slides: Idea of proof using monotonicity & localized potentials.

Monotonicity

For two conductivities $\sigma_0, \sigma_1 \in L^\infty(\Omega)$:

$$\sigma_0 \leq \sigma_1 \quad \Longrightarrow \quad \Lambda(\sigma_0) \geq \Lambda(\sigma_1)$$

This follows from

$$\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \geq \int_{\partial\Omega} g\left(\Lambda(\sigma_0) - \Lambda(\sigma_1)\right) g \geq \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)|\nabla u_0|^2$$

for all solutions u_0 of

$$\nabla \cdot (\sigma_0 \nabla u_0) = 0, \quad \sigma_0 \partial_\nu u_0|_{\partial\Omega} = g.$$

(e.g., Kang/Seo/Sheen 1997, Ikehata 1998)

Localized potentials

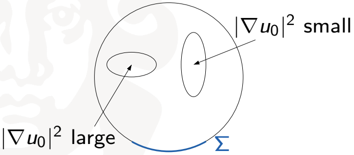
Theorem *(**H.**, IPI 2008)*

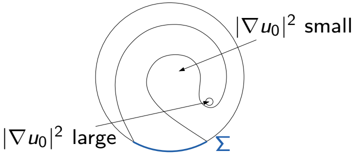
Let σ_0 fulfill unique continuation principle (UCP),

$$\overline{D_1} \cap \overline{D_2} = \emptyset, \quad \text{and} \quad \Omega \setminus (\overline{D}_1 \cup \overline{D}_2) \text{ be connected with } \Sigma.$$

Then there exist solutions $u_0^{(k)}$, $k \in \mathbb{N}$ with

$$\int_{D_1} \left|\nabla u_0^{(k)}\right|^2 \mathrm{d}x \to \infty \quad \text{and} \quad \int_{D_2} \left|\nabla u_0^{(k)}\right|^2 \mathrm{d}x \to 0.$$





Proof of shape invariance under linearization

- Linearization: $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$
- Monotonicity: For all "reference solutions" u_0:

$$\int_\Omega (\sigma - \sigma_0)|\nabla u_0|^2 \, \mathrm{d}x$$
$$\geq \underbrace{\int_{\partial\Omega} g\left(\Lambda(\sigma_0) - \Lambda(\sigma)\right) g}_{} \geq \int_\Omega \frac{\sigma_0}{\sigma}(\sigma - \sigma_0)|\nabla u_0|^2 \, \mathrm{d}x.$$
$$= -\int_{\partial\Omega} g\left(\Lambda'(\sigma_0)\kappa\right) g = \int_\Omega \kappa |\nabla u_0|^2 \, \mathrm{d}x$$

- Use localized potentials to control $|\nabla u_0|^2$

$\rightsquigarrow$ $\mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$ $\square$

Linearization and shape reconstruction

Theorem (**H.**/Seo, SIMA 2010)
Let κ, σ, σ_0 pcw. analytic.

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \quad \Longrightarrow \quad \operatorname{supp}_{\partial\Omega}\kappa = \operatorname{supp}_{\partial\Omega}(\sigma - \sigma_0)$$

$\operatorname{supp}_{\partial\Omega}$: outer support ($= \operatorname{supp} +$ parts unreachable from $\partial\Omega$)

$\rightsquigarrow$ Linearized EIT equation contains correct shape information

Can we recover conductivity changes (anomalies, inclusions, ...)
in a fast, rigorous and globally convergent way?

Monotonicity based imaging

- Monotonicity:

$$\tau \leq \sigma \quad \Longrightarrow \quad \Lambda(\tau) \geq \Lambda(\sigma)$$

- Idea: Simulate $\Lambda(\tau)$ for test cond. τ and compare with $\Lambda(\sigma)$.

  (Tamburrino/Rubinacci 02, Lionheart, Soleimani, Ventre, ...)

- Inclusion detection: For $\sigma = 1 + \chi_D$ with unknown D, use $\tau = 1 + \chi_B$, with small ball B.

$$B \subseteq D \quad \Longrightarrow \quad \tau \leq \sigma \quad \Longrightarrow \quad \Lambda(\tau) \geq \Lambda(\sigma)$$

- Algorithm: Mark all balls B with $\Lambda(1 + \chi_B) \geq \Lambda(\sigma)$
- Result: upper bound of D.

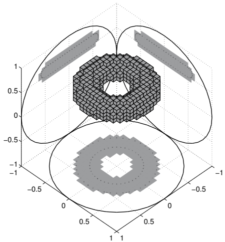
Only an upper bound? Converse monotonicity relation?

Monotonicity method (for simple test example)

Theorem *(**H.**/Ullrich, SIMA 2013)*

$\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1 + \chi_B) \geq \Lambda(\sigma).$$



For faster implementation:

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1) + \tfrac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$

Proof: Monotonicity & localized potentials

Shape can be reconstructed by linearized monotonicity tests.

$\rightsquigarrow$ fast, rigorous, allows globally convergent implementation

Improving residuum-based methods

Theorem (**H.**/Minh, preprint)

Let $\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

- Pixel partition $\Omega = \bigcup_{k=1}^{m} P_k$
- Monotonicity tests

$$\beta_k \in [0,\infty] \text{ max. values s.t. } \beta_k \Lambda'(1)\chi_{P_k} \geq \Lambda(\sigma) - \Lambda(1)$$

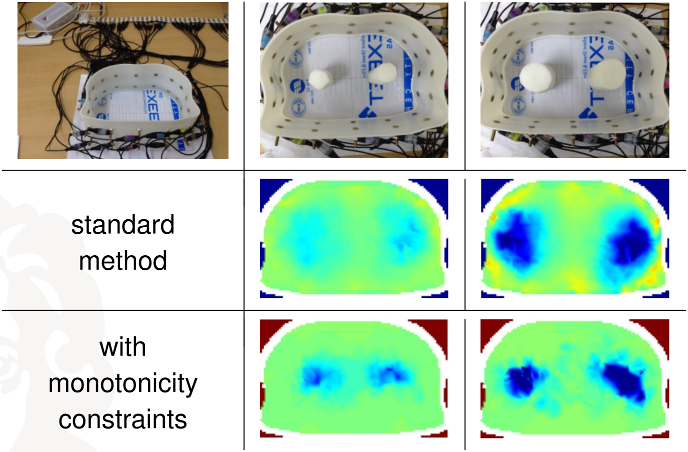
- $R(\kappa) \in \mathbb{R}^{s\times s}$: Discretization of lin. residual $\Lambda(\sigma) - \Lambda(1) - \Lambda'(1)\kappa$
  (e.g. Galerkin proj. to fin.-dim. space)

Then, the monotonicity-constrained residuum minimization problem

$$\|R(\kappa)\|_{\mathrm{F}} \to \text{min!} \quad \text{s.t.} \quad \kappa|_{P_k} = \text{const.},\ 0 \leq \kappa|_{P_k} \leq \min\{\tfrac{1}{2}, \beta_k\}$$

possesses a unique solution κ, and $P_k \subseteq \operatorname{supp}\kappa$ iff $P_k \subseteq \operatorname{supp}(\sigma - 1)$.

Phantom experiment



Enhancing standard methods by monotonicity-based constraints

(Zhou/**H.**/Seo, submitted)

Realistic data & Uncertainties

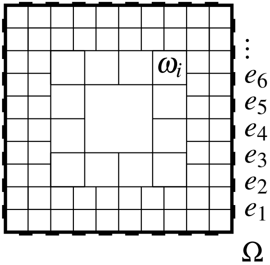
- Finite number of electrodes, CEM, noisy data $\Lambda^\delta(\sigma)$
- Unknown background, e.g., $1-\varepsilon \leq \sigma_0(x) \leq 1+\varepsilon$
- Anomaly with some minimal contrast to background, e.g.,
$$\sigma(x) = \sigma_0(x) + \kappa(x)\chi_D, \quad \kappa(x) \geq 1$$
- Can we rigorously guarantee to find inclusion D?

***H.**/Ullrich (IEEE TMI 2015):*

Rigorous Resolution Guarantee

- If $D = \emptyset$, methods return $\emptyset$.
- If $D \supset \omega_i$ then it is detected.

(Here: 32 electrodes, $\varepsilon = 1\%$, $\delta = 1.4\%$)



Conclusions

EIT is a highly ill-posed, non-linear inverse problem.

- Convergence of generic solvers unclear.
- But: Shape reconstruction in EIT is essentially a linear problem.

Monotonicity-based methods for EIT shape reconstruction

- allow fast, rigorous, globally convergent implementations.
- work in any dimensions $n \geq 2$, full or partial boundary data.
- can enhance standard residual-based methods.
- yield rigorous resolution guarantees for realistic settings.

Open problems / challenges:

- Method requires voltages on current-driven electrodes

  (***H.**, IP, to appear:* Missing electrode data may be replaced by interpolation.)
- Method applicable without definiteness, but more complicated.