Inverse problems and medical imaging

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Introduction to inverse problems

Laplace's demon

Laplace's demon: *(Pierre Simon Laplace 1814)*

"An intellect which ... would know all forces ... and all positions of all items,

if this intellect were also vast enough to submit these data to analysis ...

for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes."

Computational Science

Computational Science / Simulation Technology:

If we know all necessary parameters, then we can numerically predict the outcome of an experiment (by solving mathematical formulas).

Goals:

- Prediction
- Optimization
- Inversion/Identification

Computational Science

Generic simulation problem:

Given input x calculate outcome $y = F(x)$.

$$\begin{aligned} x \in X&: \quad \text{parameters / input} \\ y \in Y&: \quad \text{outcome / measurements} \\ F: X \to Y&: \quad \text{functional relation / model} \end{aligned}$$

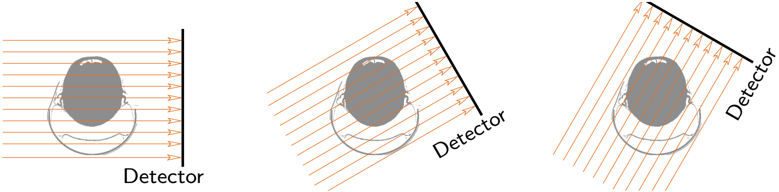
Goals:

- Prediction: Given x, calculate $y = F(x)$.
- Optimization: Find x, such that $F(x)$ is optimal.
- Inversion/Identification: Given $F(x)$, calculate x.

Example: X-ray computerized tomography (CT)

Nobel Prize in Physiology or Medicine 1979:
Allan M. Cormack and Godfrey N. Hounsfield
(Photos: Copyright ©The Nobel Foundation)

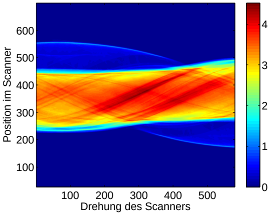
Idea: Take x-ray images from several directions



Computerized tomography (CT)

Image

$\overset{F}{\longmapsto}$



Measurements

Direct problem: Simulate/predict the measurements

(from knowledge of the interior density distribution)

Given x *calculate* $F(x) = y$*!*

Inverse problem: Reconstruct/image the interior distribution

(from taking x-ray measurements)

Given y *solve* $F(x) = y$*!*

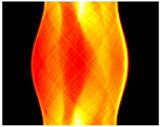
Computerized tomography

- CT forward operator $F: \ x \mapsto y$ linear

⇝ Evaluation of F is simple matrix vector multiplication
(after discretizing image and measurements as long vectors)

Simple low resolution example:

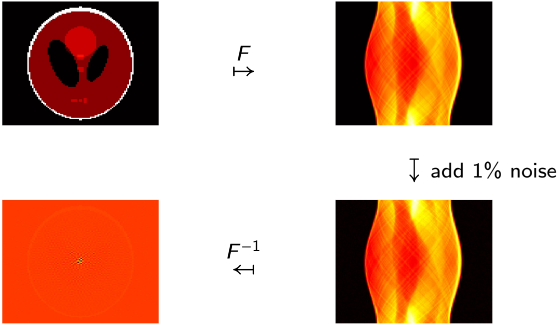
$$\overset{F}{\mapsto} \qquad \overset{F^{-1}}{\leftarrow\!\!\shortmid}$$



Problem: Matrix F invertible, but $\|F^{-1}\|$ very large.

Ill-posedness

- In the continuous case: F^{-1} not continuous
- After discretization: $\|F^{-1}\|$ very large



Are stable reconstructions impossible?

Ill-posedness

Generic linear ill-posed inverse problem

- $F: X \to Y$ bounded and linear, X, Y Hilbert spaces,
- F injective, F^{-1} not continuous,
- True solution and noise-free measurements: $F\hat{x} = \hat{y}$,
- Real measurements: y^δ with $\|y^\delta - \hat{y}\| \leq \delta$

$$F^{-1}y^\delta \not\to F^{-1}\hat{y} = \hat{x} \quad \text{for} \quad \delta \to 0.$$

Even the smallest amount of noise will corrupt the reconstructions.

Regularization

Generic linear Tikhonov regularization

$$R_\alpha = (F^* F + \alpha I)^{-1} F^*$$

$\rightsquigarrow$ R_α continuous, $R_\alpha y^\delta$ minimizes

$$\| Fx - y^\delta \|^2 + \alpha \| x \|^2 \to \text{min!}$$

Theorem. Choose $\alpha := \delta$. Then for $\delta \to 0$,

$$R_\delta y^\delta \to F^{-1} \hat{y}.$$

Regularization

Theorem. Choose $\alpha := \delta$. Then for $\delta \to 0$,

$$R_\delta y^\delta \to F^{-1}\hat{y}.$$

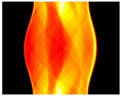
Proof. Show that $\|R_\alpha\| \leq \frac{1}{\sqrt{\alpha}}$ and apply

$$\|R_\alpha y^\delta - F^{-1}\hat{y}\| \leq \underbrace{\|R_\alpha(y^\delta - y)\|}_{\leq \|R_\alpha\|\,\delta} + \underbrace{\|R_\alpha y - F^{-1}y\|}_{\to 0 \text{ for } \alpha \to 0}.$$

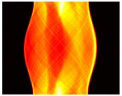
Inexact but continuous reconstruction (regularization)
+ Information on measurement noise (parameter choice rule)
= Convergence

Example

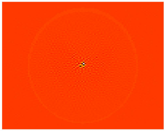
$\hat{x}$



$\hat{y} = F\hat{x}$



y^δ

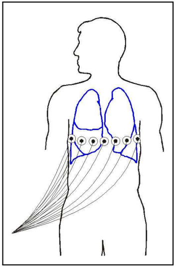
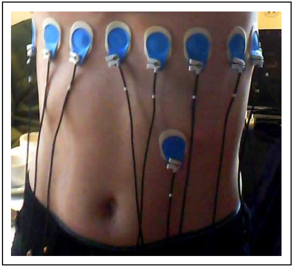
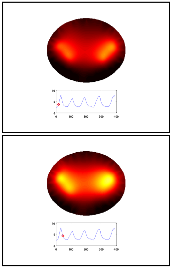


$F^{-1}y^\delta$

$(F^*F + \delta I)^{-1}F^*y^\delta$

Electrical impedance tomography

Electrical impedance tomography (EIT)

- Apply electric currents on subject's boundary
- Measure necessary voltages

⇝ Reconstruct conductivity inside subject.

Images from BMBF-project on EIT

(Hanke, Kirsch, Kress, Hahn, Weller, Schilcher, 2007-2010)

Mathematical Model

Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega$$

$\Omega \subset \mathbb{R}^n$:	imaged body, $n \geq 2$
$\sigma(x)$:	conductivity
$u(x)$:	electrical potential

Idealistic model for boundary measurements (continuum model):

$\sigma \partial_\nu u(x)\vert_{\partial\Omega}$:	applied electric current
$u(x)\vert_{\partial\Omega}$:	measured boundary voltage (potential)

Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves } (1)\}\,?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma): \; L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Challenges

EIT: Recover σ from $\Lambda(\sigma)$

- Uniqueness
  - Is σ uniquely determined from the NtD $\Lambda(\sigma)$?
- Non-linearity and ill-posedness
  - Reconstruction algorithms to determine σ from $\Lambda(\sigma)$?
  - Local/global convergence results?
- Realistic data
  - What can we recover from real measurements?
    (finite number of electrodes, realistic electrode models, ...)
  - Measurement and modelling errors? Resolution?

Next: A simple strategy (monotonicity + localized potentials) to attack these challenges.

Monotonicity

For two conductivities $\sigma_0, \sigma_1 \in L^\infty(\Omega)$:

$$\sigma_0 \leq \sigma_1 \quad \Longrightarrow \quad \Lambda(\sigma_0) \geq \Lambda(\sigma_1)$$

This follows from

$$\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \geq \int_{\partial\Omega} g\left(\Lambda(\sigma_0) - \Lambda(\sigma_1)\right) g \geq \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)|\nabla u_0|^2$$

for all solutions u_0 of

$$\nabla \cdot (\sigma_0 \nabla u_0) = 0, \quad \sigma_0 \partial_\nu u_0|_{\partial\Omega} = g.$$

(e.g., Kang/Seo/Sheen 1997, Ikehata 1998)

Can we prove uniqueness by controlling $|\nabla u_0|^2$?

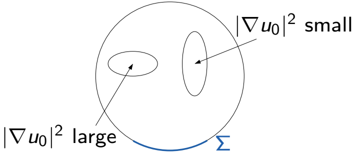
Localized potentials

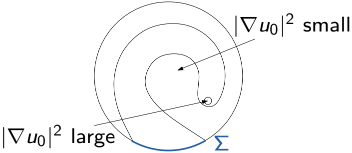
Theorem *(**H.**, 2008)*
Let σ_0 fulfill unique continuation principle (UCP),

$$\overline{D_1} \cap \overline{D_2} = \emptyset, \quad \text{and} \quad \Omega \setminus (\overline{D_1} \cup \overline{D_2}) \text{ be connected with } \partial\Omega.$$

Then there exist solutions $u_0^{(k)}$, $k \in \mathbb{N}$ with

$$\int_{D_1} \left|\nabla u_0^{(k)}\right|^2 \mathrm{d}x \to \infty \quad \text{and} \quad \int_{D_2} \left|\nabla u_0^{(k)}\right|^2 \mathrm{d}x \to 0.$$





Consequences

- Back to Calderón: Let $\Lambda(\sigma_0) = \Lambda(\sigma_1)$, σ_0 fulfills (UCP).
- By monotonicity,

$$\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x \geq 0 \geq \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x \quad \forall u_0$$

- Assume: $\exists$ neighbourhood U, connected to bndry $\partial\Omega$ where

$$\sigma_1 \geq \sigma_0 \quad \text{but} \quad \sigma_1 \not\equiv \sigma_0.$$

$\rightsquigarrow$ Potential with localized energy in U contradicts monotonicity

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Higher conductivity reachable by the bndry cannot be balanced out.

Corollary *(Druskin 1982+85, Kohn/Vogelius, 1984+85)*

Calderón problem is uniquely solvable for piecw.-anal. conductivities.

Monotonicity based imaging

- Monotonicity:

$$\tau \leq \sigma \quad \Longrightarrow \quad \Lambda(\tau) \geq \Lambda(\sigma)$$

- Idea: Simulate $\Lambda(\tau)$ for test cond. τ and compare with $\Lambda(\sigma)$.
  (Tamburrino/Rubinacci 02, Lionheart, Soleimani, Ventre, ...)
- Inclusion detection: For $\sigma = 1 + \chi_D$ with unknown D, use $\tau = 1 + \chi_B$, with small ball B.

$$B \subseteq D \quad \Longrightarrow \quad \tau \leq \sigma \quad \Longrightarrow \quad \Lambda(\tau) \geq \Lambda(\sigma)$$

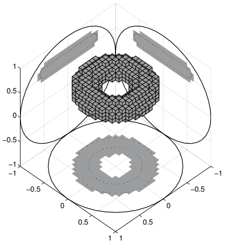
- Algorithm: Mark all balls B with $\Lambda(1 + \chi_B) \geq \Lambda(\sigma)$
- Result: upper bound of D.

Only an upper bound? Converse monotonicity relation?

Converse monotonicity relation

Theorem *(**H.**/Ullrich, 2013)*
$\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1+\chi_B) \geq \Lambda(\sigma).$$



$\rightsquigarrow$ Monotonicity method detects exact shape.

For faster implementation:

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1) + \tfrac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$

$\rightsquigarrow$ Linearized monotonicity method detects exact shape.

Proof: Monotonicity + localized potentials

Improving standard methods

- Standard EIT reconstruction methods
  - based on minimizing regularized residuum functional

$$\left\|\Lambda(\sigma) - \Lambda_{\text{measured}}\right\|^2 + \text{regularization} \to \text{min!}$$

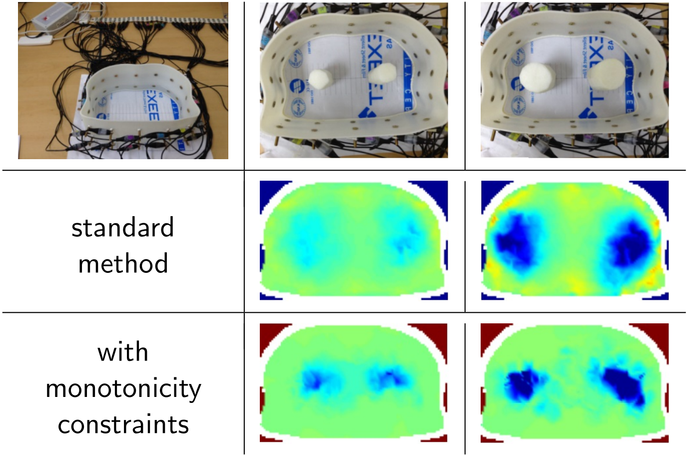
  - aim to reconstruct conductivity image $\sigma(x)$.

  Problem: No rigorous justification, no convergence result.

- Monotonicity-based inclusion/shape detection
  - rigorously justified, globally convergent,
  - only recover inclusion shape.

Can we improve standard residual-based methods using rigorous shape reconstruction methods?

Phantom experiment



Enhancing standard methods by monotonicity-based constraints

(Zhou/**H.**/Seo, submitted)

Conclusions

Computational science and inverse problems

- Computational science is the core of many new advances.
- Inverse problems is the core of new medical imaging systems.

For inverse problems in medical imaging

- Theoretical uniqueness questions are crucial and non-trivial.
- Problems are usually non-linear and ill-posed.
- Convergence of reconstruction algorithms often unknown.

For inverse coefficient problems in elliptic PDEs (such as EIT)

- Monotonicity & loc. potentials help to attack these challenges.