Inverse problems and medical imaging

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July 10, 2015

Introduction to inverse problems

Laplace's demon

Laplace's demon: *(Pierre Simon Laplace 1814)*

"An intellect which ... would know all forces ... and all positions of all items,

if this intellect were also vast enough to submit these data to analysis ...

for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes."

Computational Science

Computational Science / Simulation Technology:

If we know all necessary parameters, then we can numerically predict the outcome of an experiment (by solving mathematical formulas).

Goals:

- Prediction
- Optimization
- Inversion/Identification

Computational Science

Generic simulation problem:

Given input x calculate outcome $y = F(x)$.

$x \in X$: parameters / input
$y \in Y$: outcome / measurements
$F: X \to Y$: functional relation / model

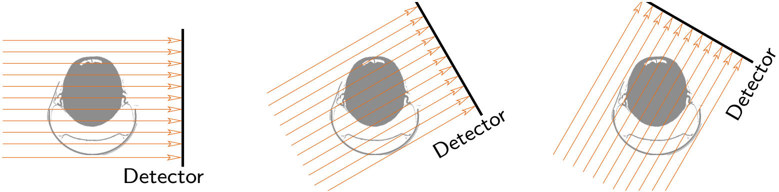
Goals:

- Prediction: Given x, calculate $y = F(x)$.
- Optimization: Find x, such that $F(x)$ is optimal.
- Inversion/Identification: Given $F(x)$, calculate x.

Example: X-ray computerized tomography (CT)

Nobel Prize in Physiology or Medicine 1979:
Allan M. Cormack and Godfrey N. Hounsfield
(Photos: Copyright ©The Nobel Foundation)

Idea: Take x-ray images from several directions

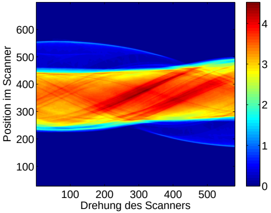


Computerized tomography (CT)

(Image: Hanke-Bourgeois, Grundlagen der Numerischen Mathematik und des Wiss. Rechnens, Teubner 2002)

Image

$\overset{F}{\longmapsto}$



Measurements

Direct problem: Simulate/predict the measurements

(from knowledge of the interior density distribution)

Given x calculate $F(x) = y$!

Inverse problem: Reconstruct/image the interior distribution

(from taking x-ray measurements)

Given y solve $F(x) = y$!

Computerized tomography

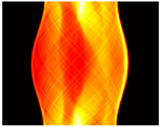
- CT forward operator $F: x \mapsto y$ is linear

$\rightsquigarrow$ Evaluation of F is simple matrix vector multiplication
(after discretizing image and measurements as long vectors)

Simple low resolution example:

$\overset{F}{\mapsto}$

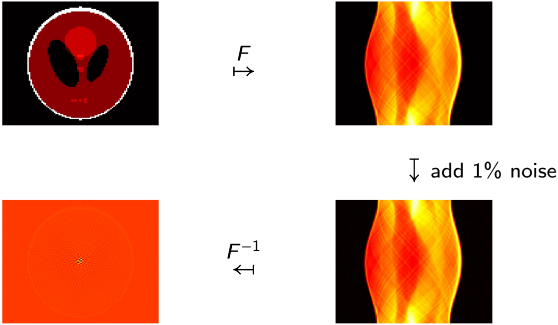
$\overset{F^{-1}}{\leftarrow\!\shortmid}$



Problem: Matrix F invertible, but $\|F^{-1}\|$ very large.

Ill-posedness

- In the continuous case: F^{-1} not continuous
- After discretization: $\|F^{-1}\|$ very large



Are stable reconstructions impossible?

Ill-posedness

Generic linear ill-posed inverse problem

- $F: X \to Y$ bounded and linear, X, Y Hilbert spaces,
- F injective, F^{-1} not continuous,
- True solution and noise-free measurements: $F\hat{x} = \hat{y}$,
- Real measurements: y^δ with $\|y^\delta - \hat{y}\| \leq \delta$

$$F^{-1}y^\delta \not\to F^{-1}\hat{y} = \hat{x} \quad \text{for} \quad \delta \to 0.$$

Even the smallest amount of noise will corrupt the reconstructions.

Regularization

Generic linear Tikhonov regularization

$$R_\alpha = (F^* F + \alpha I)^{-1} F^*$$

$\rightsquigarrow$ R_α continuous, $R_\alpha y^\delta$ minimizes

$$\| Fx - y^\delta \|^2 + \alpha \| x \|^2 \to \min!$$

Theorem. Choose $\alpha := \delta$. Then for $\delta \to 0$,

$$R_\delta y^\delta \to F^{-1} \hat{y}.$$

Regularization

Theorem. Choose $\alpha := \delta$. Then for $\delta \to 0$,

$$R_\delta y^\delta \to F^{-1}\hat{y}.$$

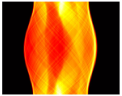
Proof. Show that $\|R_\alpha\| \leq \frac{1}{\sqrt{\alpha}}$ and apply

$$\|R_\alpha y^\delta - F^{-1}\hat{y}\| \leq \underbrace{\|R_\alpha(y^\delta - y)\|}_{\leq \|R_\alpha\|\delta} + \underbrace{\|R_\alpha y - F^{-1}y\|}_{\to 0 \text{ for } \alpha \to 0}.$$

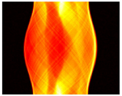
Inexact but continuous reconstruction (regularization)
+ Information on measurement noise (parameter choice rule)
= Convergence

Example

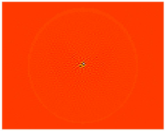
$\hat{x}$



$\hat{y} = F\hat{x}$



y^δ

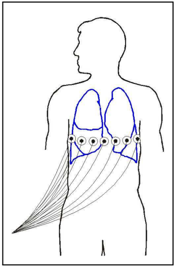


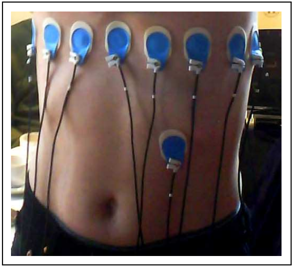
$F^{-1}y^\delta$

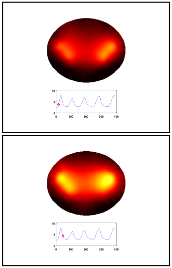
$(F^*F + \delta I)^{-1}F^*y^\delta$

Electrical impedance tomography

Electrical impedance tomography (EIT)







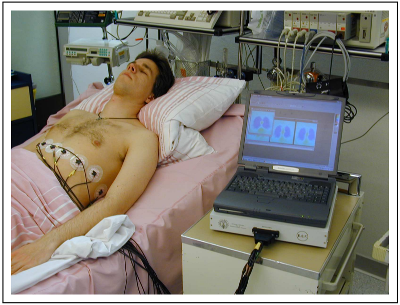
- Apply electric currents on subject's boundary
- Measure necessary voltages

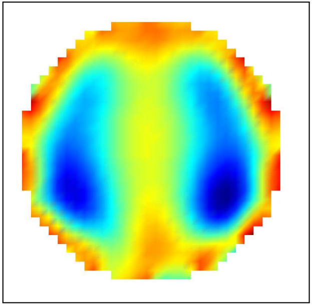
⇝ Reconstruct conductivity inside subject.

Images from BMBF-project on EIT

(Hanke, Kirsch, Kress, Hahn, Weller, Schilcher, 2007-2010)

MF-System Goe-MF II





Electric current strength: $5 - 500\mathrm{mA}_{\mathrm{rms}}$, 44 images/second,
CE certified by Viasys Healthcare, approved for clinical research

Mathematical Model

Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega$$

$\Omega \subset \mathbb{R}^n$: imaged body, $n \geq 2$
$\sigma(x)$: conductivity
$u(x)$: electrical potential

Idealistic model for boundary measurements (continuum model):

$\sigma \partial_\nu u(x)|_{\partial\Omega}$: applied electric current
$u(x)|_{\partial\Omega}$: measured boundary voltage (potential)

Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves } (1)\}\,?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma): \ L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Inversion of $\sigma \mapsto \Lambda(\sigma) = \Lambda_{\text{meas}}$?

Generic solvers for non-linear inverse problems:

- Linearize and regularize:

  $$\Lambda_{\text{meas}} = \Lambda(\sigma) \approx \Lambda(\sigma_0) + \Lambda'(\sigma_0)(\sigma - \sigma_0).$$

  σ_0: Initial guess or reference state (e.g. exhaled state)

  $\rightsquigarrow$ Linear inverse problem for σ

  (Solve, e.g., using linear Tikhonov regul., repeat for Newton-type algorithm.)

- Regularize and linearize:
  E.g., minimize non-linear Tikhonov functional

  $$\|\Lambda_{\text{meas}} - \Lambda(\sigma)\|^2 + \alpha \|\sigma - \sigma_0\|^2 \to \text{min!}$$

Generic and flexible, but high comput. cost and convergence unclear

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(PhD-project of Dominik Garmatter: Reduce comput. costs by model reduction)

Linearization and shape reconstruction

Theorem (**H.**/Seo, SIAM J. Math. Anal. 2010)
Let κ, σ, σ_0 pcw. analytic.

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \quad \Longrightarrow \quad \mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$$

$\mathrm{supp}_{\partial\Omega}$: outer support ($= \mathrm{supp}$ + parts unreachable from $\partial\Omega$)

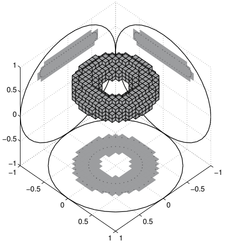
- Linearized EIT equation contains correct shape information
- For the shape reconstruction problem

$$\Lambda(\sigma) \mapsto \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$$

fast, rigorous and globally convergent method seem possible.

Theorem heavily inspired by Factorization Method (Kirsch/Hanke/Brühl 1998/99) which is fast and rigorous (but for which convergence is unclear).

Monotonicity method (for simple test example)



Theorem *(**H.**/Ullrich, 2013)*
$\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1 + \chi_B) \geq \Lambda(\sigma).$$

For faster implementation:

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1) + \tfrac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$

Shape can be reconstructed by linearized monotonicity tests.

$\rightsquigarrow$ fast, rigorous, allows globally convergent implementation

Improving residuum-based methods

Theorem (**H.**/Minh, preprint)
Let $\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

- Pixel partition $\Omega = \bigcup_{k=1}^{m} P_k$
- Monotonicity tests

$$\beta_k \in [0, \infty] \text{ max. values s.t. } \beta_k \Lambda'(1)\chi_{P_k} \geq \Lambda(\sigma) - \Lambda(1)$$

- $R(\kappa) \in \mathbb{R}^{s\times s}$: Discretization of lin. residual $\Lambda(\sigma) - \Lambda(1) - \Lambda'(1)\kappa$
  (e.g. Galerkin proj. to fin.-dim. space)

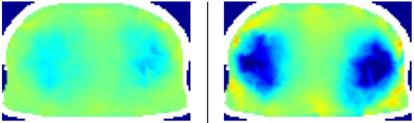
Then, the monotonicity-constrained residuum minimization problem

$$\|R(\kappa)\|_F \to \min! \quad \text{s.t.} \quad \kappa|_{P_k} = \text{const.}, \; 0 \leq \kappa|_{P_k} \leq \min\{\tfrac{1}{2}, \beta_k\}$$

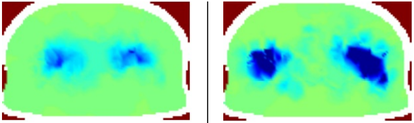
possesses a unique solution κ, and $P_k \subseteq \operatorname{supp} \kappa$ iff $P_k \subseteq \operatorname{supp}(\sigma - 1)$.

Phantom experiment

standard method



with monotonicity constraints



Enhancing standard methods by monotonicity-based constraints

(Zhou/**H.**/Seo, submitted)

Conclusions

Computational science and inverse problems

- Computational science is the core of many new advances.
- Inverse problems is the core of new medical imaging systems.

For ill-posed inverse problems

- Regularization is required for convergent algorithms.
- Regularization can also incorporate additional information (e.g., total variation penalization, stochastic priors, etc.)

For the non-linear ill-posed inverse problem of EIT

- Convergence of standard regul. techniques is still unclear.
- Monotonicity-based regularization allow fast, rigorous, and globally convergent reconstruction of shape information.