Inverse problems and medical imaging

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Introduction to inverse problems

Pierre Simon Laplace (1814):

*"An intellect which ... would know
all forces ... and all positions of all items,*

*if this intellect were also vast enough to
submit these data to analysis ...*

*for such an intellect nothing would be
uncertain and the future just like the past
would be present before its eyes."*

Computational Science / Simulation Technology:

If we know all necessary parameters, then we can numerically predict the outcome of an experiment (by solving mathematical formulas).

Goals:

- Prediction
- Optimization
- Inversion/Identification

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Generic simulation problem:

Given input x calculate outcome $y = F(x)$.

$\quad x \in X$: parameters / input

$\quad y \in Y$: outcome / measurements

$F : X \to Y$: functional relation / model

Goals:

- Prediction: Given x, calculate $y = F(x)$.

- Optimization: Find x, such that $F(x)$ is optimal.

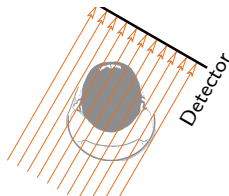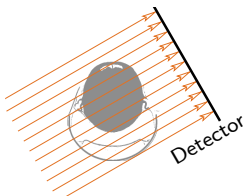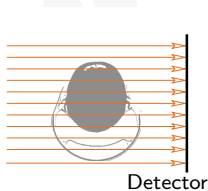- Inversion/Identification: Given $F(x)$, calculate x.

Example: X-ray computerized tomography (CT)

Nobel Prize in Physiology or Medicine 1979:
Allan M. Cormack and Godfrey N. Hounsfield
(Photos: Copyright ©The Nobel Foundation)

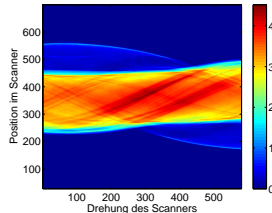Idea: Take x-ray images from several directions

Computerized tomography (CT)

(Image: Hanke-Bourgeois, Grundlagen der Numerischen Mathematik und des Wiss. Rechnens, Teubner 2002)

$$\overset{F}{\longmapsto}$$

Image   Measurements

Direct problem: Simulate/predict the measurements

(from knowledge of the interior density distribution)

Given x calculate $F(x) = y$!

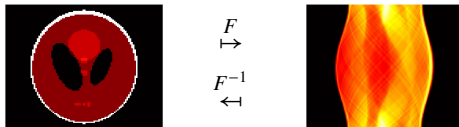Inverse problem: Reconstruct/image the interior distribution

(from taking x-ray measurements)

Given y solve $F(x) = y$!

- ‣ CT forward operator $F : x \mapsto y$ is linear
- ⤳ Evaluation of F is simple matrix vector multiplication
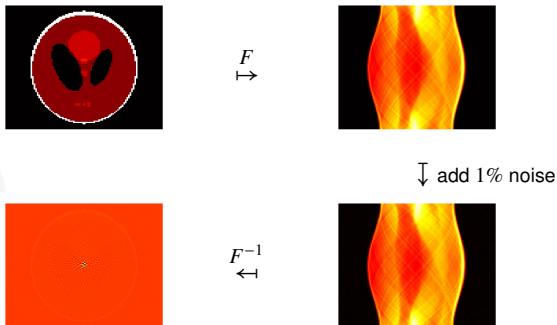 (after discretizing image and measurements as long vectors)

Simple low resolution example:

$\overset{F}{\mapsto}$

$\overset{F^{-1}}{\longleftarrow}$

Problem: Matrix F invertible, but $\|F^{-1}\|$ very large.

Ill-posedness

- In the continuous case: F^{-1} not continuous
- After discretization: $\|F^{-1}\|$ very large

$F \atop \mapsto$

$\downarrow$ add 1% noise

$F^{-1} \atop \hookleftarrow$

Are stable reconstructions impossible?

Generic linear ill-posed inverse problem

- $F : X \to Y$ bounded and linear, X, Y Hilbert spaces,
- F injective, F^{-1} not continuous,
- True solution and noise-free measurements: $F\hat{x} = \hat{y}$,
- Real measurements: y^δ with $\|y^\delta - \hat{y}\| \le \delta$

$$F^{-1}y^\delta \not\to F^{-1}\hat{y} = \hat{x} \quad \text{for} \quad \delta \to 0.$$

Even the smallest noise may corrupt the reconstructions.

Generic linear Tikhonov regularization

$$R_\alpha = \left(F^*F + \alpha I\right)^{-1} F^*$$

$\rightsquigarrow R_\alpha$ continuous, $R_\alpha y^\delta$ minimizes

$$\left\| Fx - y^\delta \right\|^2 + \alpha \left\| x \right\|^2 \to \min!$$

Theorem. Choose $\alpha := \delta$. Then for $\delta \to 0$,

$$R_\delta y^\delta \to F^{-1} \hat{y}.$$

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Theorem. Choose $\alpha := \delta$. Then for $\delta \to 0$,

$$R_\delta y^\delta \to F^{-1}\hat{y}.$$

Proof. Show that $\|R_\alpha\| \leq \frac{1}{\sqrt{\alpha}}$ and apply

$$\|R_\alpha y^\delta - F^{-1}\hat{y}\| \leq \underbrace{\|R_\alpha(y^\delta - y)\|}_{\leq \|R_\alpha\| \, \delta} + \underbrace{\|R_\alpha y - F^{-1}y\|}_{\to 0 \text{ for } \alpha \to 0}.$$

Inexact but continuous reconstruction (regularization)
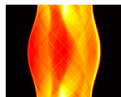
+ Information on measurement noise (parameter choice rule)

= Convergence

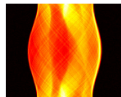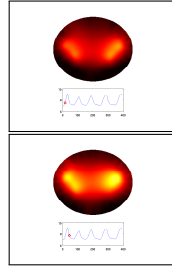$\hat{x}$ $\hat{y} = F\hat{x}$ y^{δ}
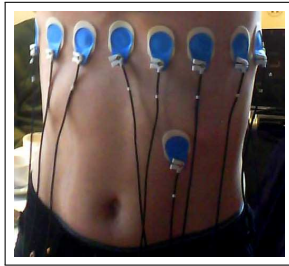
$F^{-1}y^{\delta}$

$(F^{*}F + \delta I)^{-1}F^{*}y^{\delta}$

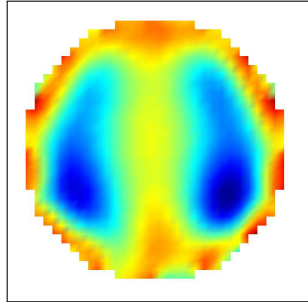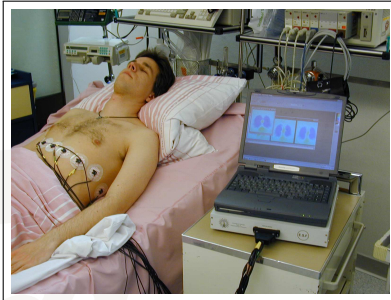# Electrical impedance tomography
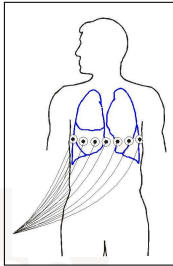
- ▸ Apply electric currents on subject's boundary
- ▸ Measure necessary voltages
- ↝ Reconstruct conductivity inside subject.

Images from BMBF-project on EIT

(Hanke, Kirsch, Kress, Hahn, Weller, Schilcher, 2007-2010)

Electric current strength: $5 - 500\mathrm{mA_{rms}}$, 44 images/second,
CE certified by Viasys Healthcare, approved for clinical research

- Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega \qquad \text{(EIT)}$$

  $\Omega \subset \mathbb{R}^n$:   imaged body, $n \geq 2$

  $\sigma(x)$:   conductivity

  $u(x)$:   electrical potential

- Idealistic model for boundary meas. (continuum model):

  $\sigma \partial_\nu u(x)|_{\partial\Omega}$:   applied electric current

  $u(x)|_{\partial\Omega}$:   measured boundary voltage (potential)
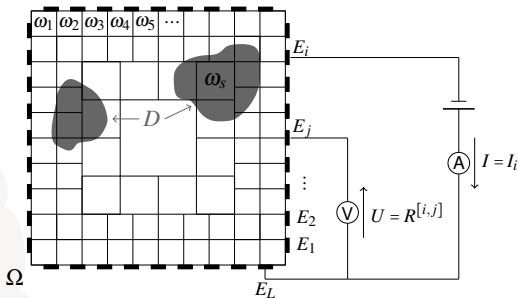
- Neumann-to-Dirichlet-Operator:

$$\Lambda(\sigma): \ L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

  where u solves (EIT) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Realistic electrode model

▸ Complete electrode model including contact impedances z_l
  (for $z_l = 0$: shunt electrode model)

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \Omega$$
$$u|_{E_j} + z_l \sigma \partial_\nu u|_{E_j} = \text{const.} =: R^{[i,j]}(\sigma)$$
$$\sigma \partial_\nu u|_{\partial \Omega} = 0 \quad \text{outside } E_l$$
$$\int_{E_l} \sigma \partial_\nu u \, ds = I(\delta_{i,l} - \delta_{j,L})$$

Roughly speaking, $R(\sigma) \to \Lambda(\sigma)$ for #electrodes $\to \infty$

Inverse problem of EIT: Recover σ from $\Lambda(\sigma)$, resp., $R(\sigma)$

Challenges:

- Uniqueness
    - Is σ uniquely determined from "perfect data" $\Lambda(\sigma)$?
- Non-linearity and ill-posedness
    - Reconstruction algorithms to determine σ from $\Lambda(\sigma)$?
    - Local/global convergence results when using noisy $R(\sigma)$?
- Realistic data
    - What can we recover from real measurements?
      (fixed number of electrodes, realistic electrode models, ...)
    - Measurement and modelling errors? Resolution?

Generic solvers for non-linear inverse problems:

- Linearize and regularize:

$$\Lambda_{meas} = \Lambda(\sigma) \approx \Lambda(\sigma_0) + \Lambda'(\sigma_0)(\sigma - \sigma_0).$$

  σ_0: Initial guess or reference state (e.g. exhaled state)

  $\rightsquigarrow$ Linear inverse problem for σ

  (Solve, e.g., using linear Tikhonov regul., repeat for Newton-type algorithm.)

- Regularize and linearize:

  E.g., minimize non-linear Tikhonov functional

$$\|\Lambda_{meas} - \Lambda(\sigma)\|^2 + \alpha \|\sigma - \sigma_0\|^2 \to min!$$

Very flexible, but high comput. cost and convergence unclear

Theorem (**H**./Seo, SIAM J. Math. Anal. 2010)
Let κ, σ, σ_0 pcw. analytic.

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \quad \Longrightarrow \quad \operatorname{supp}_{\partial\Omega}\kappa = \operatorname{supp}_{\partial\Omega}(\sigma - \sigma_0)$$

$\operatorname{supp}_{\partial\Omega}$: outer support (= supp + parts unreachable from $\partial\Omega$)

- Linearized EIT equation contains correct shape information
- For the shape reconstruction problem

$$\Lambda(\sigma) \mapsto \operatorname{supp}_{\partial\Omega}(\sigma - \sigma_0)$$
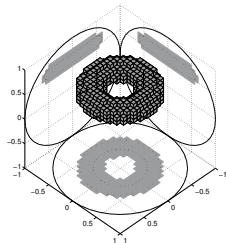
fast, rigorous and globally convergent method seem possible.

Monotonicity method (for simple test example)

Theorem *(H./Ullrich, SIAM J. Math. Anal. 2013)*

$\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1 + \chi_B) \geq \Lambda(\sigma).$$

For faster implementation:

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1) + \tfrac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$

Shape can be reconstructed by linearized monotonicity tests.

$\rightsquigarrow$ fast, rigorous, allows globally convergent implementation

For real electrode measurements $R(\sigma)$:

"$\Longrightarrow$": still holds

"$\Longleftarrow$": holds if "enough" electrodes are used

Theorem (**H.**/Minh, preprint)

Let $\Omega \smallsetminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

- Pixel partition $\Omega = \bigcup_{k=1}^m P_k$
- Monotonicity tests

$$\beta_k \in [0, \infty] \text{ max. values s.t. } \beta_k \Lambda'(1)\chi_{P_k} \geq \Lambda(\sigma) - \Lambda(1)$$
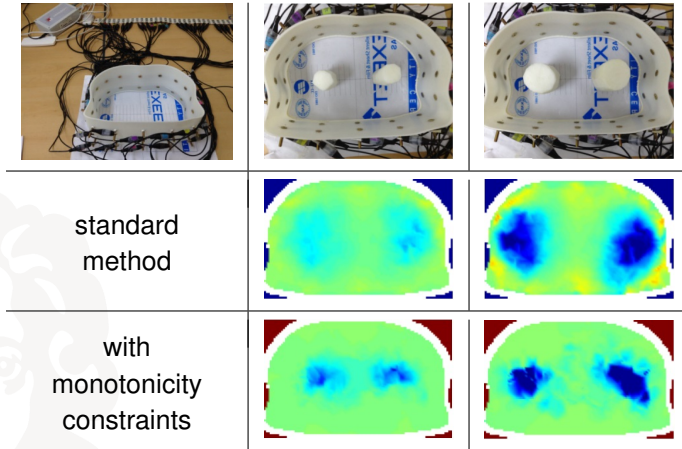
- $L(\kappa) \in \mathbb{R}^{s \times s}$: Discretization of lin. residual $\Lambda(\sigma) - \Lambda(1) - \Lambda'(1)\kappa$
  (e.g. Galerkin proj. to fin.-dim. space)

Then, the monotonicity-constrained residuum minimization problem

$$\|L(\kappa)\|_\mathsf{F} \to \min! \quad \text{s.t.} \quad \kappa|_{P_k} = \text{const.}, \ 0 \leq \kappa|_{P_k} \leq \min\{\tfrac{1}{2}, \beta_k\}$$

possesses a unique solution κ, and $P_k \subseteq \operatorname{supp} \kappa$ iff $P_k \subseteq \operatorname{supp}(\sigma - 1)$.

Phantom experiment

Enhancing standard methods by monotonicity-based constraints

(Zhou/**H.**/Seo, submitted)

Computational science and inverse problems

- ▸ Computational science is the core of many new advances.
- ▸ Inverse problems is the core of new medical imaging systems.

For ill-posed inverse problems

- ▸ Regularization is required for convergent algorithms.
- ▸ Regularization can also incorporate additional information
  (e.g., total variation penalization, stochastic priors, etc.)

For the non-linear ill-posed inverse problem of EIT

- ▸ Convergence of standard regul. techniques is still unclear.
- ▸ Monotonicity-based regularization allow fast, rigorous, and globally convergent reconstruction of shape information.