Introduction to the numerical solution of inverse problems and shape detection in electrical impedance tomography

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0 Introduction and preliminaries

These lecture notes summarize the theorems and proofs for my four lectures given at the Advanced Instructional School on Theoretical and Numerical Aspects of Inverse Problems. The slides of the lecture can be found on my webpage

`http://www.mathematik.uni-stuttgart.de/oip`

We summarize some notations and preliminaries.

Definition and Theorem 0.1. Let X and Y be real Hilbert spaces with scalar products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$ and corresponding norms

$$\|x\|_X := \sqrt{(x,x)_X}, \quad \text{and} \quad \|y\|_Y := \sqrt{(y,y)_Y}.$$

For a linear operator $A: \; X \to Y$, the following two statements are equivalent:

(a) A is continuous.

(b) A is bounded, i.e., there exists a constant $C > 0$ with

$$\|Ax\| \leq C \|x\| \quad \forall x \in X.$$

The space of continuous linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. We also write $\mathcal{L}(X) := \mathcal{L}(X,X)$.

Definition and Theorem 0.2.

(a) $\mathcal{L}(X,Y)$ is a Banach spaces with respect to the norm

$$\|A\|_{\mathcal{L}(X,Y)} := \sup_{x\neq 0} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{\|x\|_X=1} \|Ax\|_Y .$$

(b) For $A \in \mathcal{L}(X,Y)$, there exists a unique operator $A^* \in \mathcal{L}(Y,X)$ that fulfills

$$(x, A^*y)_X := (Ax, y)_Y \quad \forall x \in X, y \in Y.$$

$A*$ is called the **adjoint operator** or A. An operator $A \in \mathcal{L}(X)$ with $A = A^*$ is called **self-adjoint**. For self-adjoint operators A, it holds that

$$\|A\|_{\mathcal{L}(X,Y)} = \sup_{x\neq 0} \frac{(x,Ax)_X}{\|x\|_X^2} = \sup_{\|x\|_X=1} (x,Ax)_X.$$

Theorem 0.3. (Lax-Milgram)
Let X be a Hilbert space and

$$b : X \times X \to \mathbb{R},$$

be a continuous, symmetric and coercive bilinear form, i.e.

$$\begin{aligned} b(u,v) &= b(v,u) && \forall u,v \in X && \text{(Symmetry)}\\ \exists C>0\ :\ |b(u,v)| &\le C\,\|u\|\,\|v\| && \forall u,v \in X && \text{(Continuity)}\\ \exists \beta>0\ :\ b(u,u) &\ge \beta\,\|u\|^2 && \forall u \in X && \text{(Coercivity)} \end{aligned}$$

and b is linear in each of its two arguments. Let $l \in \mathcal{L}(X,\mathbb{R})$.

Then there exists a unique $u \in X$ with

$$b(u,v) = l(v) \quad \text{für alle } v \in X.$$

u depends linearly and continuous on l,

$$\|u\|_X \le \frac{1}{\beta}\,\|l\|_{\mathcal{L}(X,\mathbb{R})}\,.$$

1 Theorems and proofs for Lecture 1

Definition 1.1. A linear operator $F \in \mathcal{L}(X,Y)$ is called **compact**, if $\overline{F(U)}$ is compact for alle bounded $U \subseteq X$, i.e. if $(x_n)_{n\in\mathbb{N}} \subset X$ is a bounded sequence then $(F(x_n))_{n\in\mathbb{N}} \subset Y$ contains a bounded subsequence.

Theorem 1.2. If F is compact and injective, and $\dim X = \infty$, then F^{-1} is not continuous, i.e., the inverse problem $Fx = y$ is ill-posed.

Proof. Since $\dim X = \infty$, we can choose an infinite sequence of orthonormal vectors $(x_n)_{n\in\mathbb{N}}$. From the orthonormality, it follows that

$$\|x_n - x_m\| = \sqrt{2} \quad \forall n,m \in \mathbb{N},\ n \neq m,$$

so that no subsequence of $(x_n)_{n\in\mathbb{N}}$ can be convergent.

On the other hand, $(x_n)_{n\in\mathbb{N}}$ is bounded, and F is compact, so that $(Fx_n)_{n\in\mathbb{N}}$ must contain a converging subsequence $(Fx_{n_k})_{k\in\mathbb{N}}$. Hence, $(F(x_{n_k}))_{k\in\mathbb{N}}$ converges but $(x_{n_k})_{k\in\mathbb{N}}$ does not converge, which shows that F^{-1} cannot be continuous. $\square$

Theorem 1.3. Every limit of compact operators is compact.

Proof. Let $(F_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X,Y)$ be a sequence of compact operators, and let F be a bounded linear operator with

$$\|F_n - F\|_{\mathcal{L}(X,Y)} \to 0.$$

We will show that F is compact, i.e. that the image of every bounded sequence has a converging subsequence.

Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a bounded sequence. Then,

- there exists a subsequence $(x_{1,l})_{l\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}} \subset X$, so that $F_1 x_{1,l}$ converges,
- there exists a (sub-)subsequence $(x_{2,l})_{l\in\mathbb{N}}$ of $(x_{1,l})_{l\in\mathbb{N}}$, so that $F_2 x_{2,l}$ converges,

Proceeding this way, we obtain a nested subsequences $(x_{k,l})_{l\in\mathbb{N}}$ so that $F_k x_{k,l}$ converges. Now consider the diagonal sequence $(x_{l,l})_{l\in\mathbb{N}}$. For each $k \in \mathbb{N}$, the sequence $(F_k x_{l,l})_{l\in\mathbb{N}}$ convergences. Hence, for all $k, l, m \in \mathbb{N}$

$$
\begin{aligned}
&\|F x_{l,l} - F x_{m,m}\| \\
&\le \|F x_{l,l} - F_k x_{l,l}\| + \|F_k x_{l,l} - F_k x_{m,m}\| + \|F_k x_{m,m} - F x_{m,m}\| \\
&\le \|F - F_k\| \left(\|x_{l,l}\| + \|x_{m,m}\|\right) + \|F_k x_{l,l} - F_k x_{m,m}\|
\end{aligned}
$$

The first term becomes arbitrarily small for sufficiently large k, the second term becomes arbitrarily small for sufficiently large $l(k), m(k)$. Hence,

$$\lim_{l,m\to\infty} \|F x_{l,l} - F x_{m,m}\| = 0,$$

so that $(F x_{l,l})_{l\in\mathbb{N}}$ is a Cauchy-sequence and thus convergent. $\square$

Theorem 1.4. If $F \in \mathcal{L}(X, Y)$ and $\dim \mathcal{R}(F) < \infty$ then F is compact.

Proof. Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a bounded sequence. Then $(F x_n)_{n\in\mathbb{N}}$ is a bounded sequence in the finite-dimensional space $\mathcal{R}(F)$, and in finite dimensions, every bounded sequence contains a converging subsequence (Theorem of Bolzano-Weierstrass). $\square$

Theorem 1.5. Let $F \in \mathcal{L}(X, Y)$ possess an unbounded left inverse F^{-1}, and let $R_n \in \mathcal{L}(Y, X)$ be a sequence with

$$R_n y \to F^{-1} y \quad \text{for all } y \in \mathcal{R}(F).$$

Then $\|R_n\| \to \infty$.

Proof. Assume that there exists a constant $C > 0$ such that

$$\|R_{n_k}\| \le C$$

for a subsequence $(R_{n_k})_{k\in\mathbb{N}}$. Then for all $y \in \mathcal{R}(F)$

$$\left\|F^{-1} y\right\| = \lim_{k\to\infty} \|R_{n_k} y\| \le C \|x\|,$$

so that F^{-1} is bounded. Hence, the assertion follows by contraposition. $\square$

2 Theorems and proofs for Lecture 2

Theorem 2.1. Let $A \in \mathcal{L}(X,Y)$. For each $\alpha > 0$, the operators

$$A^*A + \alpha I \in \mathcal{L}(X) \quad \text{and} \quad AA^* + \alpha I \in \mathcal{L}(Y)$$

are continuously invertible, and they fulfill

$$\left\|(A^*A + \alpha I)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{1}{\alpha}, \quad \left\|(AA^* + \alpha I)^{-1}\right\|_{\mathcal{L}(Y)} \leq \frac{1}{\alpha}.$$

Proof. Let $z \in X$. A vector $x \in X$ solves

$$(A^*A + \alpha I)x = z$$

if and only if it solves

$$b(x,\xi) := ((A^*A + \alpha I)x, \xi)_X = (z,\xi)_X =: l(\xi) \quad \forall \xi \in X.$$

$b : \; X \times X \to \mathbb{R}$ is bilinear and fulfills

$$b(x,\xi) = (Ax, A\xi)_X + \alpha (x,\xi)_X,$$

so that b is symmetric, continuous and coercive with coercivity constant α. $l \in \mathcal{L}(X,\mathbb{R})$ fulfills $\|l\|_{\mathcal{L}(X,\mathbb{R})} \leq \|z\|_X$.
Hence, it follows from the Theorem of Lax-Milgram (Theorem 0.3) that $A^*A + \alpha I$ is continuously invertible with

$$\left\|(A^*A + \alpha I)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{1}{\alpha}.$$

The same arguments prove the assertion for $AA^* + \alpha I$. □

Theorem 2.2. Let $A \in \mathcal{L}(X,Y)$. $x_\alpha^\delta := (A^*A + \alpha I)^{-1} A^* y^\delta$ is the unique minimizer of the Tikhonov functional

$$J_\alpha(x) := \left\|Ax - y^\delta\right\|_Y^2 + \alpha \left\|x\right\|_X^2$$

Proof. Let $x_\alpha^\delta := (A^*A + \alpha I)^{-1} A^* y^\delta$. For all $x \in X$, $x \neq x_\alpha^\delta$, we have that

$$\begin{aligned}
&\left\|y^\delta - Ax\right\|_Y^2 + \alpha \left\|x\right\|_X^2 - \left\|y^\delta - Ax_\alpha^\delta\right\|_Y^2 + \alpha \left\|x_\alpha^\delta\right\|_X^2 \\
&= -2(y^\delta, Ax)_Y + (x, (A^*A + \alpha I)x)_X + 2(y^\delta, Ax_\alpha^\delta)_Y - (x_\alpha^\delta, (A^*A + \alpha I)x_\alpha^\delta)_X \\
&= \left((x - x_\alpha^\delta), (A^*A + \alpha I)(x - x_\alpha^\delta)\right)_X + 2\left(x, (A^*A + \alpha I)x_\alpha^\delta\right)_X - 2(y^\delta, Ax)_Y \\
&\qquad - 2\left(x_\alpha^\delta, (A^*A + \alpha I)x_\alpha^\delta\right)_X + 2(y^\delta, Ax_\alpha^\delta)_Y \\
&= \left\|A(x - x_\alpha^\delta)\right\|_Y^2 + \alpha \left\|x - x_\alpha^\delta\right\|_X^2 > 0.
\end{aligned}$$

This shows that $J_\alpha(x) > J_\alpha(x_\alpha^\delta)$ for all $x \neq x_\alpha^\delta$. □

Theorem 2.3. Let $A \in \mathcal{L}(X, Y)$ be injective.

(a) $\mathcal{R}(A^*A)$ is dense in X

(b) If a sequence $(x_k)_{k\in\mathbb{N}} \subset X$ fulfills

$$A^*Ax_k \to A^*Ax \quad \text{and} \quad \|x_k\|_X \leq \|x\|_X$$

then $x_k \to x$.

Proof. (a) Assume that $\mathcal{R}(A^*A)$ was not dense in X. Then there would exist a vector $x \neq 0$ with $x \in \mathcal{R}(A^*A)^\perp$, i.e.,

$$(x, A^*A\xi) = 0 \quad \text{for all } \xi \in X.$$

Choosing $\xi = x$ we would obtain that $\|Ax\|^2 = (x, A^*Ax) = 0$, which would contradict the injectivity of A.

(b) For every $z \in Y$ we have that

$$\begin{aligned} |(x, x_k - x)_X| &= |(A^*Az, x_k - x)_X| + |(x - A^*Az, x_k - x)_X| \\ &\leq \|z\|_Y \|A^*A(x_k - x)\|_Y + 2\|x - A^*Az\|_X \|x\|_X \end{aligned}$$

Since the range of A^*A is dense in X, the second summand can be made arbitrarily small by choosing an appropriate z. The first summand becomes arbitrarily small for sufficiently large k. Hence,

$$(x, x_k - x) \to 0.$$

This shows that $(x, x_k) \to \|x\|_X^2$. Using the assumption $\|x_k\|_X \leq \|x\|_X$ it follows that

$$\|x - x_k\|_X^2 = \|x\|_X^2 - 2(x, x_k)_X + \|x_k\|_X^2 \leq 2\|x\|_X^2 - 2(x, x_k) \to 0,$$

so that $x_k \to x$. $\square$

Theorem 2.4. Let $A \in \mathcal{L}(X, Y)$ be injective (with a possibly unbounded left inverse). Let $A\hat{x} = \hat{y}$ and let $(y^\delta)_{\delta>0} \subseteq Y$ be noisy measurements with $\left\|y^\delta - \hat{y}\right\|_Y \leq \delta$.

If we choose the regularization parameter so that

$$\alpha(\delta) \to 0 \quad \text{and} \quad \frac{\delta^2}{\alpha(\delta)} \to 0,$$

then

$$(A^*A + \alpha I)^{-1}A^*y^\delta \to \hat{x} \quad \text{for } \delta \to 0.$$

Proof. Writing

$$R_\alpha := (A^*A + \alpha I)^{-1}A^* \quad \text{and} \quad x_\alpha^\delta := R_\alpha y^\delta$$

we have that

$$\left\|x_\alpha^\delta - \hat{x}\right\|_X \le \left\|R_\alpha(y^\delta - \hat{y})\right\|_X + \|R_\alpha \hat{y} - \hat{x}\|_X .$$

We will show that (a) $\|R_\alpha\| \le \frac{1}{\sqrt{\alpha}}$ and (b) $R_\alpha \hat{y} \to \hat{x}$ for $\alpha \to 0$.

(a) One easily checks that

$$R_\alpha = (A^*A + \alpha I)^{-1}A^* = A^*(AA^* + \alpha I)^{-1},$$

and that $(A^*A + \alpha I)^{-1}$, $(AA^* + \alpha I)^{-1}$ and AR_α are self-adjoint operators.

For all $y \in Y$, it follows that

$$(AR_\alpha y, y)_Y = (y, y) - \alpha\left((AA^* + \alpha I)^{-1}y, y\right) \le \|y\|_Y^2 ,$$

which shows that $\|AR_\alpha\| \le 1$.

Using Theorem 2.1 we obtain that

$$\begin{aligned} \|R_\alpha\|_{\mathcal{L}(Y,X)}^2 &= \sup_{y \in Y,\ \|y\|_Y = 1} \|R_\alpha y\|_X^2 = \|R_\alpha^* R_\alpha\|_{\mathcal{L}(Y)} \\ &\le \left\|(AA^* + \alpha I)^{-1}\right\|_{\mathcal{L}(Y)} \|AR_\alpha\|_{\mathcal{L}(Y)} \le \frac{1}{\alpha}. \end{aligned}$$

Hence, $\|R_\alpha\| \le \frac{1}{\sqrt{\alpha}}$.

(b) We write $x_\alpha := R_\alpha \hat{y}$. Using the minimizer property of x_α in Theorem 2.2 we obtain that

$$\|\hat{y} - Ax_\alpha\|^2 + \alpha \|x_\alpha\|^2 \le \|\hat{y} - A\hat{x}\|^2 + \alpha \|\hat{x}\|^2 ,$$

and thus $\|x_\alpha\| \le \|\hat{x}\|$.

Furthermore, we have that

$$\begin{aligned} \alpha \|\hat{x}\|_X^2 &= \|(A^*A + \alpha I)(x_\alpha - \hat{x})\|_X^2 \\ &= \|A^*A(x_\alpha - \hat{x})\|_X^2 + 2\alpha \|A(x_\alpha - \hat{x})\|_Y + \alpha^2 \|x_\alpha - \hat{x}\|_X^2 . \end{aligned}$$

Since x_α is bounded, it follows that

$$A^*A(x_\alpha - \hat{x}) \to 0,$$

and, together with $\|x_\alpha\| \le \|\hat{x}\|$, we obtain from Theorem 2.3 that $x_\alpha \to \hat{x}$. $\square$

3 Theorems and proofs for Lecture 3

Definition 3.1. Let $d \in \mathbb{R}^n$, $|d| = 1$ be an arbitrary direction. Let Φ_z solve

$$\Delta \Phi_z = d \cdot \nabla \delta_z \text{ in } \Omega, \quad \partial_\nu \Phi_z|_{\partial\Omega} = 0$$

and $\int_{\partial\Omega} \Phi_z \, \mathrm{d}s = 0$. ($\Phi_z$ is called **dipole function**).

Definition 3.2. We define the **virtual measurements**

$$L_D : \; L^2_\diamond(D)^n \to L^2_\diamond(\partial\Omega), \quad F \mapsto v|_\partial\Omega,$$

where $v \in H^1_\diamond(\Omega)$ solves

$$\int_\Omega \nabla v \cdot \nabla w \, \mathrm{d}x = \int_D F \cdot \nabla w \, \mathrm{d}x \quad \forall w \in H^1_\diamond(\Omega).$$

Theorem 3.3. For all unit vectors $d \in \mathbb{R}^n$, $\|d\| = 1$, and every point $z \in \Omega \setminus \partial D$,

$$z \in D \quad \text{ if and only if } \quad \Phi_z|_{\partial\Omega} \in \mathcal{R}(L_D).$$

Proof. (a) First let $z \in D$ and $\epsilon > 0$ be such that $\overline{B_\epsilon(z)} \subseteq D$. We choose

$$\begin{aligned} &f_1 \in H^1(B_\epsilon(z)) \quad \text{with} \quad f_1|_{\partial B_\epsilon(z)} = \Phi_{z,d}|_{\partial B_\epsilon(z)} \\ &f_2 \in H^1(B_\epsilon(z)) \quad \text{with} \quad \Delta f_2 = 0, \quad \text{and} \quad \partial_\nu f_2|_{\partial B_\epsilon(z)} = \partial_\nu \Phi_{z,d}|_{\partial B_\epsilon(z)}. \end{aligned}$$

and define $F \in L^2(D)^n$ as the zero continuation of $\nabla(f_1 - f_2) \in L^2(B_\epsilon(z))^n$ to D. Then the function

$$v := \begin{cases} \Phi_{z,d} & \text{in } \Omega \setminus \overline{B_\epsilon(z)} \\ f_1 & \text{in } B_\epsilon(z). \end{cases}$$

fulfills $v \in H^1_\diamond(\Omega)$ and, for all $w \in H^1_\diamond(\Omega)$,

$$\begin{aligned} \int_\Omega \nabla v \cdot \nabla w \, \mathrm{d}x &= \int_{\Omega \setminus \overline{B_\epsilon(z)}} \nabla \Phi_{z,d} \cdot \nabla w \, \mathrm{d}x + \int_{B_\epsilon(z)} \nabla f_1 \cdot \nabla w \, \mathrm{d}x \\ &= - \int_{\partial B_\epsilon(z)} \partial_\nu \Phi_{z,d} \, w|_{\partial B_\epsilon(z)} \, \mathrm{d}s + \int_{B_\epsilon(z)} \nabla f_1 \cdot \nabla w \, \mathrm{d}x \\ &= \int_{B_\epsilon(z)} \nabla (f_1 - f_2) \cdot \nabla w \, \mathrm{d}x = \int_D F \cdot \nabla w \, \mathrm{d}x. \end{aligned}$$

This shows that $\Phi_{z,d}|_{\partial\Omega} = v|_{\partial\Omega} = L_D(F) \in \mathcal{R}(L_D)$.

(b) Now let $\Phi_{z,d}|_\Sigma \in \mathcal{R}(L_D)$. Let $v \in H^1_\diamond(\Omega)$ be the function from the definition of L_D. Then

$$v|_{\partial\Omega} = \Phi_{z,d}|_{\partial\Omega} \quad \text{and} \quad \partial_\nu v|_{\partial\Omega} = 0 = \partial_\nu \Phi_{z,d}|_{\partial\Omega}.$$

It follows by unique continuation that $v = \Phi_{z,d}$ in the connected set $\Omega \setminus (\overline{D} \cup \{z\})$. If $z \notin \overline{D}$ then $d \cdot \nabla \delta_z \notin H^{-2}(\Omega \setminus \overline{D})$, and thus $\Phi_{z,d} \notin L^2(\Omega \setminus \overline{D})$, which contradicts that $v = \Phi_{z,d}$ in $\Omega \setminus (\overline{D} \cup \{z\})$. Hence, $z \in \overline{D}$. $\square$

Theorem 3.4. The adjoint operator of L_D is given by

$$L_D^*: \ L^2_\diamond(\partial\Omega) \to L^2(D)^n, \qquad g \mapsto \nabla u_0|_D,$$

where $u_0 \in H^1_\diamond(\Omega)$ solves

$$\Delta u_0 = 0 \text{ in } \Omega, \quad \text{and} \qquad \partial_\nu u_0|_{\partial\Omega} = g.$$

Proof. For all $g \in L^2_\diamond(\partial\Omega)$, and $F \in L^2(D)^n$, we have that,

$$\begin{aligned}\int_D (L_D^* g) \cdot F \,\mathrm{d}x &= \int_{\partial\Omega} g(L_D F) \,\mathrm{d}x = \int_{\partial\Omega} g v|_{\partial\Omega} \,\mathrm{d}x \\ &= \int_\Omega \nabla u_0 \cdot \nabla v \,\mathrm{d}x = \int_D \nabla u_0 \cdot F \,\mathrm{d}x,\end{aligned}$$

which shows the assertion. □

Theorem 3.5. Let $\sigma_1, \sigma_0 \in L^\infty_+(\Omega)$. Then, for all $g \in L^2_\diamond(\partial\Omega)$,

$$\begin{aligned}\int_\Omega (\sigma_0 - \sigma_1)|\nabla u_0|^2 \,\mathrm{d}x \ &\leq \ \int_{\partial\Omega} g\,(\Lambda(\sigma_1) - \Lambda(\sigma_0))\, g \,\mathrm{d}s \\ &\leq \ \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_0 - \sigma_1)|\nabla u_0|^2 \,\mathrm{d}x,\end{aligned}$$

where $u_0 \in H^1_\diamond(\Omega)$ solves $\nabla \cdot (\sigma_0 \nabla u_0) = 0$ in Ω, and $\sigma_0 \partial_\nu u_0|_{\partial\Omega} = g$.

Proof. Let $g \in L^2_\diamond(\partial\Omega)$, and let $u_1 \in H^1_\diamond(\Omega)$ solve $\nabla\cdot(\sigma_1 \nabla u_1) = 0$ in Ω, and $\sigma_1 \partial_\nu u_1|_{\partial\Omega} = g$.

Then,

$$\int_\Omega \sigma_1 \nabla u_1 \cdot \nabla u_0 \,\mathrm{d}x = \int_\Omega g u_0 \,\mathrm{d}s = \int_\Omega \sigma_0 \nabla u_0 \cdot \nabla u_0 \,\mathrm{d}x = \int_{\partial\Omega} g \Lambda_0 g \,\mathrm{d}s.$$

Hence, using

$$\begin{aligned}&\int_\Omega \sigma_1 \nabla(u_1 - u_0) \cdot \nabla(u_1 - u_0) \,\mathrm{d}x \\ &= \int_\Omega \sigma_1 |\nabla u_1|^2 \,\mathrm{d}x - \int_\Omega \sigma_0 |\nabla u_0|^2 \,\mathrm{d}x + \int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x,\end{aligned}$$

we obtain that

$$\int_{\partial\Omega} g(\Lambda(\sigma_1) - \Lambda(\sigma_0)) g \,\mathrm{d}s = \int_\Omega (\sigma_0 - \sigma_1)|\nabla u_0|^2 \,\mathrm{d}x + \int_\Omega \sigma_1 |\nabla(u_1 - u_0)|^2 \,\mathrm{d}x,$$

which already yields the first asserted inequality.

By interchanging σ_1 and σ_0 we conclude

$$\begin{aligned}\int_{\partial\Omega} g(\Lambda(\sigma_0) - \Lambda(\sigma_1)) g \,\mathrm{d}s &= \int_\Omega (\sigma_1 - \sigma_0)|\nabla u_1|^2 \,\mathrm{d}x + \int_\Omega \sigma_0 |\nabla(u_0 - u_1)|^2 \,\mathrm{d}x \\ &= \int_\Omega \left(\sigma_1 \left| \nabla u_1 - \frac{\sigma_0}{\sigma_1} \nabla u_0 \right|^2 + \left(\sigma_0 - \frac{\sigma_0^2}{\sigma_1} \right) |\nabla u_0|^2, \right) \mathrm{d}x\end{aligned}$$

and hence obtain the second inequality. □

Theorem 3.6. Let X_1, X_2, and Y be three real Hilbert spaces, $A_i \in \mathcal{L}(X_i, Y)$, $i = 1, 2$. If there exists $C > 0$ with

$$\|A_1^* y\|_{X_1} \leq C \|A_2^* y\|_{X_2} \quad \text{for all } y \in Y$$

then $\mathcal{R}(A_1) \subseteq \mathcal{R}(A_2)$.

Proof. Let $y \in \mathcal{R}(A_1)$. Then there exists $x \in Y$ such that $y = A_1 x$ and thus

$$|(y, \eta)_Y| = |(A_1 x, \eta)_Y| = |(x, A_1^* \eta)_X| \leq \|x\|_X \|A_1^* \eta\|_{X_1} \leq C \|x\|_X \|A_2^* \eta\|_{X_2} \ \forall \eta \in Y.$$

This shows that the linear functional

$$l(\xi) := (y, \eta)_Y \quad \text{for every } \xi = A_2^* \eta \in \mathcal{R}(A_2^*) \subseteq X.$$

fulfills $|l(\xi)| \leq C \|x\| \|\xi\|$. Hence, l is well-defined and continuous on $\mathcal{R}(A_2^*)$. Using the Riesz theorem, it follows that there exists $x' \in X$ with

$$(x', \xi)_Y = l(\xi) \quad \text{for all } \xi \in \mathcal{R}(A_2^*).$$

Hence, for all $\eta \in Y$ we have that

$$(A_2 x', \eta) = (x', A_2^* \eta) = l(A_2^* \eta) = (y, \eta),$$

and hence $y = A_2 x' \in \mathcal{R}(A_2)$. □

4 Theorems and proofs for Lecture 4

Theorem 4.1. Let

- D be open, $\overline{D} \subset \Omega$, and $\Omega \setminus \overline{D}$ connected,
- B be open, $\overline{B} \subset \Omega$, and $B \not\subseteq D$.

Then there exists $(g_m)_{m \in \mathbb{N}} \subset L^2_\diamond(\partial\Omega)$ s.t. the solutions $(u_m)_{m \in \mathbb{N}}$ of

$$\Delta u_m = 0 \quad \text{in } \Omega, \qquad \partial_\nu u_m|_{\partial\Omega} = g_m,$$

fulfill

$$\lim_{m \to \infty} \int_B |\nabla u_m|^2 \, \mathrm{d}x = \infty \quad \text{and} \quad \lim_{m \to \infty} \int_D |\nabla u_m|^2 \, \mathrm{d}x = 0.$$

Proof. (a) Reformulation as range (non-)inclusion.
Let L_D, L_B be the *virtual measurement* operators for the sets D and B from Definition 3.2. By Theorem 3.4, the adjoint operators

$$L_D^* : \ L^2_\diamond(\partial\Omega) \to L^2(D)^n \quad \text{and} \quad L_B^* : \ L^2_\diamond(\partial\Omega) \to L^2(B)^n$$

fulfill $L_B^* g = \nabla u|_B$ and $L_D^* g = \nabla u|_D$, where $u \in H^1_\diamond(B)$ solves

$$\Delta u = 0 \quad \text{and} \quad \partial_\nu u|_{\partial\Omega} = g.$$

Hence, the assertion is equivalent to the statement

$$\nexists C > 0 \ : \ \|L_B^* g\| \leq C \, \|L_D^* g\| \quad \forall g \in L^2_\diamond(\partial\Omega)$$

which follows from Theorem 3.6 if we can show that the range (non-)inclusion

$$\mathcal{R}(L_B) \not\subseteq \mathcal{R}(L_D). \tag{1}$$

(b) Proof of the range (non-)inclusion.
Since $B \not\subseteq D$, and both, B and D, are open, there exists $z \in B$ with $z \notin \overline{D}$. Let Φ_z be the dipole function from Definition 3.1 (with an arbitrary direction d). Then we obtain from Theorem 3.3 that

$$\Phi_z \in \mathcal{R}(L_B) \quad \text{and} \quad \Phi_z \notin \mathcal{R}(L_D),$$

which shows that $\mathcal{R}(L_B) \not\subseteq \mathcal{R}(L_D)$. □

Theorem 4.2. Let

- $\sigma(x) = 1 + \chi_D(x)$, with
- D open, $\overline{D} \subset \Omega$, and $\Omega \setminus \overline{D}$ connected.

Then for each open ball $B \subseteq \Omega$,

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1) + \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma)$$

Proof. Throughout the proof, $u \in H^1_\diamond(\Omega)$ denotes the solution of

$$\Delta u = 0 \quad \text{and} \quad \partial_\nu u|_{\partial\Omega} = g.$$

First let $B \subseteq D$. Using the monotonicity result in Theorem 3.5, we obtain that

$$\begin{aligned}\int_{\partial\Omega} g\,(\Lambda(\sigma) - \Lambda(1))\, g \,\mathrm{d}s &\leq \int_\Omega \frac{1}{\sigma}(1-\sigma)|\nabla u|^2 \,\mathrm{d}x = -\int_D \frac{1}{2}|\nabla u|^2 \,\mathrm{d}x \\ &\leq -\int_B \frac{1}{2}|\nabla u|^2 \,\mathrm{d}x = \frac{1}{2}\int_{\partial\Omega} g\,(\Lambda'(1)\chi_B)\, g \,\mathrm{d}s,\end{aligned}$$

which shows that $\Lambda(\sigma) - \Lambda(1) \leq \frac{1}{2}\Lambda'(1)\chi_B$.
Now let $B \not\subseteq D$. Again using the monotonicity result in Theorem 3.5, we obtain that

$$\begin{aligned}\int_{\partial\Omega} g\left(\Lambda(\sigma) - \Lambda(1) - \frac{1}{2}\Lambda'(1)\chi_B\right) g \,\mathrm{d}s &\geq \int_\Omega (1-\sigma)|\nabla u|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\partial\Omega} g\,(\Lambda'(1)\chi_B)\, g \,\mathrm{d}s \\ &= -\int_D |\nabla u|^2 \,\mathrm{d}x + \int_B \frac{1}{2}|\nabla u|^2 \,\mathrm{d}x.\end{aligned}$$

Using the localized potentials from Theorem 5.1, we obtain a g for which

$$-\int_D |\nabla u|^2 \,\mathrm{d}x + \int_B \frac{1}{2}|\nabla u|^2 \,\mathrm{d}x \geq 0,$$

which shows that $\Lambda(\sigma) - \Lambda(1) - \frac{1}{2}\Lambda'(1)\chi_B \not\leq 0$. □