# Inverse parameter identification problems

Bastian von Harrach

harrach@math.uni-stuttgart.de

Chair of Optimization and Inverse Problems, University of Stuttgart, Germany

ICP - Institute for Computational Physics,
June 27, 2013.

# Contents

- Parameter identification problems
  in diffuse optical tomography (DOT)
- Linearized reconstruction algorithms
  in electrical impedance tomography (EIT)

# Parameter identification problems

in diffuse optical tomography (DOT)

# Diffuse optical tomography

Diffuse optical tomography (DOT):

- Transilluminate biological tissue with visible/near-infrared light
- Goal: Reconstruct spatial image of interior physical properties.

  Relevant quantities (in diffusive regime):
  - Scattering
  - Absorption
- Applications:
  - Breast cancer detection
  - Bedside-imaging of neonatal brain function

# Diffuse optical tomography

Figure 1 from Gibson, Hebden and Arridge (Phys. Med. Biol. 50, R1–R43, 2005)
Imaging Diagnostic System Inc.'s computed tomography laser breast imaging system.
See www.imds.com for more details.

# Mathematical Model

- General Forward Model:
  Photon transport models (Boltzmann transport equation)
  cf., e.g., Bal, Inverse Problems 25, 053001 (48pp), 2009.
- For highly scattering media:
  - DC diffusion approximation for photon density $u$:

$$-\nabla \cdot (a\nabla u) + cu = 0 \quad \text{in } B \subset \mathbb{R}^n,$$

| | | |
|---|---|---|
| $u:$ | $B \to \mathbb{R}$: | photon density |
| $a:$ | $B \to \mathbb{R}$: | diffusion/scattering coefficient |
| $c:$ | $B \to \mathbb{R}$: | absorption coefficient |

  - Boundary measurements (idealized):

Neumann and Dirichlet data $u|_S$, $a\partial_\nu u|_S$ on $S \subseteq \partial B$.

Remaining boundary assumed to be insulated, $a\partial_\nu u|_{\partial B \setminus \overline{S}} = 0$.

# Forward & inverse problem

DC diffuse optical tomography:

$$-\nabla \cdot (a \nabla u) + cu = 0 \quad \text{in } B \subset \mathbb{R}^n,\ n \geq 2,$$

$B$ bounded with smooth bndry, $S \subseteq \partial B$ open part, $a, c \in L^\infty_+(B)$.

- $\forall g \in L^2(S)\ \exists!$ sol. $u \in H^1(B)$ : $a\partial_\nu u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{on } B \setminus \overline{S}. \end{cases}$
- (Local) Neumann-to-Dirichlet map

$$\Lambda_{a,c} : \ g \mapsto u|_S, \quad L^2(S) \to L^2(S)$$

is linear, compact and self-adjoint.

---

Inverse Problem: Can we reconstruct $a$ and $c$ from $\Lambda_{a,c}$?

---

# Non-uniqueness

DC diffuse optical tomography:

$$-\nabla \cdot (a\nabla u) + cu = 0$$

Arridge/Lionheart (1998 Opt. Lett. 23 882–4):

- $v := \sqrt{a}u$ solves

$$-\Delta v + \eta v = 0, \quad \text{with} \quad \eta = \frac{\Delta\sqrt{a}}{\sqrt{a}} + \frac{c}{a}.$$

- $a = 1$ around $S$ $\quad\rightsquigarrow\quad$ $(u|_S, a\partial_\nu u|_S) = (v|_S, \partial_\nu v|_S)$.

$\rightsquigarrow$ $\Lambda_{a,c}$ only depends on effective absorption $\eta = \eta(a,c)$.

---

Absorption and scattering effects cannot be distinguished.

---

(Note: Argument requires smooth scattering coefficient $a$).

# Experimental results

Theory: Absorption and scattering effects cannot be distinguished.

Practice:

Successful separate reconstructions of absorption and scattering (from phantom experiment using dc diffusion model!)

Pei et al. (2001), Jiang et al. (2002), Schmitz et al. (2002), Xu et al. (2002)

$\rightsquigarrow$ Practice contradicts theory!

Pei et al. (2001):

*"As a matter of established methodological principle (...) empirical facts have the right-of-way; if a theoretical derivation yields a conclusion that is at odds with experimental results, the reconciliatory burden falls on the theorist, not on the experimentalist.“*

# New uniqueness result

**Theorem** (**H.**, Inverse Problems 2009)

- $a_1, a_2 \in L^\infty_+(B)$ piecewise constant
- $c_1, c_2 \in L^\infty_+(B)$ piecewise analytic

If $\Lambda_{a_1,c_1} = \Lambda_{a_2,c_2}$ then $a_1 = a_2$ and $c_1 = c_2$.

- Piecewise constantness seems fulfilled for phantom experiments.

$\rightsquigarrow$ Result reconciles theory with practice.

- Measurements contain more than just the effective absorption!

Next slides: Idea of the proof using monotony and loc. potentials.

# Monotony result

## Lemma

Let $a_1, a_2, c_1, c_2 \in L^\infty_+(B)$. Then for all $g \in L^2(S)$,

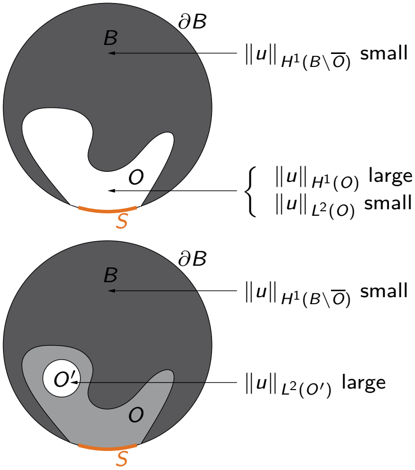
$$\int_B \left( (a_2 - a_1)|\nabla u_1|^2 + (c_2 - c_1)|u_1|^2 \right) \, \mathrm{d}x$$
$$\geq \langle (\Lambda_{a_1,c_1} - \Lambda_{a_2,c_2}) g, g \rangle$$
$$\geq \int_B \left( (a_2 - a_1)|\nabla u_2|^2 + (c_2 - c_1)|u_2|^2 \right) \, \mathrm{d}x,$$

$u_1, u_2 \in H^1(B)$: solutions for $(a_1, c_1)$, resp., $(a_2, c_2)$.

Can we control $|u_j|^2$ and $|\nabla u_j|^2$?

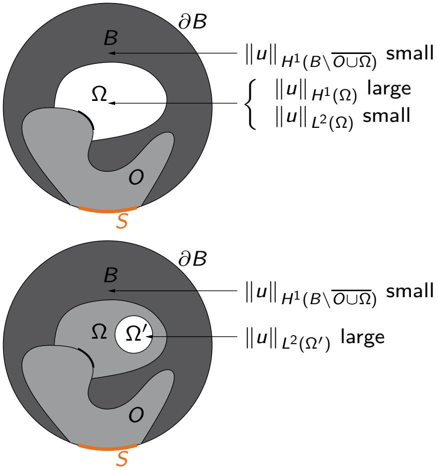
# Localized potentials

Lemma There exist solutions $u$ with



# Localized potentials

Lemma There exist solutions $u$ with



# Proof of uniqueness

## Monotony

$$\begin{aligned}\int_B \left((a_2 - a_1)|\nabla u_1|^2 + (c_2 - c_1)|u_1|^2\right)\,\mathrm{d}x \\ \geq \langle(\Lambda_{a_1,c_1} - \Lambda_{a_2,c_2})g, g\rangle \\ \geq \int_B \left((a_2 - a_1)|\nabla u_2|^2 + (c_2 - c_1)|u_2|^2\right)\,\mathrm{d}x,\end{aligned}$$

## Proof of the uniqueness result (*very sketchy ...*)

Start with region next to $S$

- Use loc. pot. with $|\nabla u|^2 \to \infty$ in that region $\rightsquigarrow a_1 = a_2$
- Then use loc. pot. with $|u|^2 \to \infty$ in that region $\rightsquigarrow c_1 = c_2$
- Repeat over all regions.

# Uniqueness or not?

- Arridge/Lionheart (1998): Non-uniqueness for general smooth $(a, c)$.
- **H.** (2009): Uniqueness for piecew. constant $a$, piecew. analytic $c$.

## What information about $(a, c)$ does $\Lambda_{a,c}$ really contain?

Formally(!), $\Lambda_{a,c}$ can only determine $\eta = \frac{\Delta\sqrt{a}}{\sqrt{a}} + \frac{c}{a}$.

Jumps in $a$ or $\nabla a$ $\rightsquigarrow$ distributional singularities in $\Delta\sqrt{a}$.

Bold guess: Maybe $\Lambda_{a,c}$ determines

- $\eta$ where $a$ and $c$ are smooth,
- jumps in $a$ and $\nabla a$.

(However, note that $\Delta\sqrt{a}/\sqrt{a}$ is not well-defined for non-smooth $a$ ...)

# Exact characterization

Theorem ( **H.**, Inverse Probl. Imaging 2012)

Let $a_1, a_2, c_1, c_2 \in L^\infty_+(B)$ piecewise analytic on joint partition

$$B = O_1 \cup \ldots \cup O_J \cup \Gamma, \qquad \partial O_1 \cup \ldots \cup \partial O_J = \partial B \cup \Gamma.$$

Then, $\Lambda_{a_1,c_1} = \Lambda_{a_2,c_2}$ if and only if

(a) $\quad a_1|_S = a_2|_S, \quad$ and $\quad \partial_\nu a_1|_S = \partial_\nu a_2|_S \quad$ on $S$,

(b) $$\frac{\partial_\nu a_1}{a_1}\Big|_{\partial B\setminus \overline{S}} = \frac{\partial_\nu a_2}{a_2}\Big|_{\partial B\setminus \overline{S}} \quad \text{on } \partial B \setminus \overline{S},$$

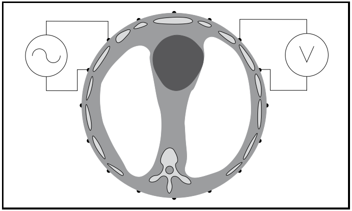
(c) $$\eta_1 := \frac{\Delta\sqrt{a_1}}{\sqrt{a_1}} + \frac{c_1}{a_1} = \frac{\Delta\sqrt{a_2}}{\sqrt{a_2}} + \frac{c_2}{a_2} =: \eta_2 \quad \text{on } B \setminus \Gamma,$$

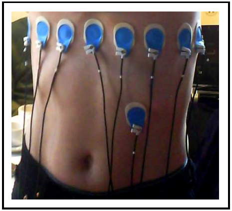
(d) $$\frac{a_1^+|_\Gamma}{a_1^-|_\Gamma} = \frac{a_2^+|_\Gamma}{a_2^-|_\Gamma}, \quad \text{and} \quad \frac{[\partial_\nu a_2]_\Gamma}{a_2^-|_\Gamma} = \frac{[\partial_\nu a_1]_\Gamma}{a_1^-|_\Gamma} \quad \text{on } \Gamma.$$

# Linearized reconstruction algorithms

in electrical impedance tomography (EIT)

# EIT



Electrical impedance tomography (EIT):

- Apply currents $\sigma \partial_\nu u|_{\partial B}$ (Neumann boundary data)
  - $\rightsquigarrow$ Electric potential $u$ solves

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } B$$

- Measure voltages $u|_{\partial B}$ (Dirichlet boundary data)

Current-Voltage-Measurements $\rightsquigarrow$ Neumann-to-Dirichlet map $\Lambda(\sigma)$

# Inverse problem

Non-linear forward operator of EIT

$$\Lambda : \ \sigma \mapsto \Lambda(\sigma), \quad L^\infty_+(B) \to \mathcal{L}(L^2_\diamond(\partial B))$$

Inverse problem of EIT: $\Lambda(\sigma) \mapsto \sigma?$

Localized potentials $\leadsto$ Uniqueness for piecew. analytic conductivities

already known: Druskin (1982+85), Kohn/Vogelius (1984+85)

# Linearization

Generic approach: Linearization

$$\Lambda(\sigma) - \Lambda(\sigma_0) \approx \Lambda'(\sigma_0)(\sigma - \sigma_0)$$

$\sigma_0$: known reference conductivity / initial guess / ...

$\Lambda'(\sigma_0)$: Fréchet-Derivative / sensitivity matrix.

$$\Lambda'(\sigma_0) : L^\infty_+(B) \to \mathcal{L}(L^2_\diamond(\partial B)).$$

$\rightsquigarrow$ Solve linearized equation for difference $\sigma - \sigma_0$.

Often: $\mathrm{supp}(\sigma - \sigma_0) \subset\subset B$ compact. *("shape" / "inclusion")*

# Linearization

## Linear reconstruction method

e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

Solve $\Lambda'(\sigma_0)\kappa \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, then $\kappa \approx \sigma - \sigma_0$.

- Multiple possibilities to measure residual norm and to regularize.
- No rigorous theory for single linearization step.
- Almost no theory for Newton iteration:
  - Dobson (1992): (Local) convergence for regularized EIT equation.
  - Lechleiter/Rieder(2008): (Local) convergence for discretized setting.
  - No (local) convergence theory for non-discretized case!

# Linearization

## Linear reconstruction method

e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

Solve $\Lambda'(\sigma_0)\kappa \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, then $\kappa \approx \sigma - \sigma_0$.

- Seemingly, no rigorous results possible for single lineariz. step.
- Seemingly, only justifiable for small $\sigma - \sigma_0$ (local results).

Here: Rigorous and global(!) result about the linearization error.

# Exact Linearization

**Theorem** (**H.**/Seo, SIAM J. Math. Anal. 2010)

Let $\kappa$, $\sigma$, $\sigma_0$ piecewise analytic and $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$. Then

$$\mathrm{supp}_{\partial B}\kappa = \mathrm{supp}_{\partial B}(\sigma - \sigma_0)$$

$\mathrm{supp}_{\partial B}$: outer support ( $=$ support, if support is compact and has conn. complement)

- <u>Exact solution</u> of lin. equation yields correct (outer) shape.
- No assumptions on $\sigma - \sigma_0$!

$\rightsquigarrow$ Linearization error does not lead to shape errors.

Proof: Combination of monotony and localized potentials.

# Proof

- Exact linearization: $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$
- Monotony: For all "reference solutions" $u_0$:

$$\int_B (\sigma - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x$$

$$\geq \underbrace{\langle g, (\Lambda(\sigma) - \Lambda(\sigma_0))\, g\rangle}_{= \int_B \kappa |\nabla u_0|^2 \,\mathrm{d}x} \geq \int_B \frac{\sigma_0}{\sigma}(\sigma - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x.$$

- Use localized potentials to control $|\nabla u_0|^2$

$\rightsquigarrow$ $\mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$

# Interpretation

$\sigma_0$: reference conductivity
$\sigma$ : true conductivity

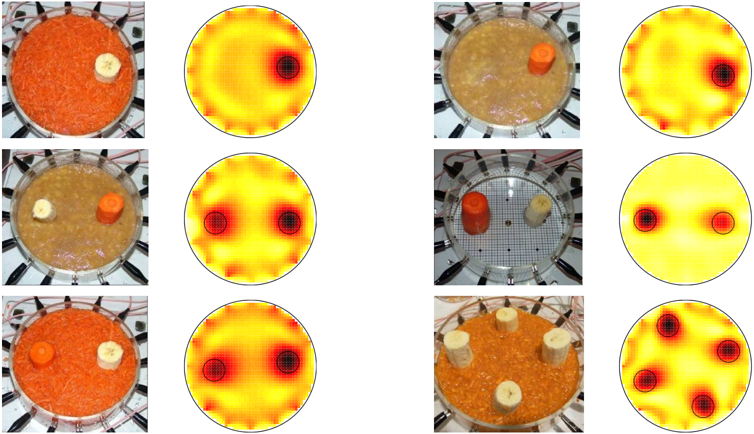
- Current paths depend on unknown conductivity $\sigma$ (Non-linearity of EIT)
- Linearizing EIT around ref. conductivity $\sigma_0$ corresponds to assuming that currents take the same paths as in a ref. body.
- Paths may differ considerably when $\sigma - \sigma_0$ is large.
- However, taking the (wrong) reference paths for reconstruction still yields the correct shape information!

Practical issues: Theorem requires exact sol. of linearized equation

- Does an approximate solution of the linearized equation show approximately correct shapes?

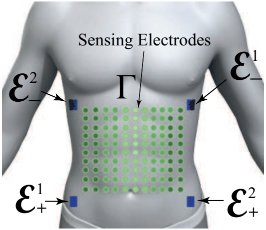
# Experimental result (frequency difference data)

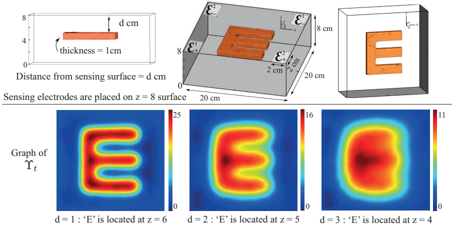
**H.**/*Seo/Woo (IEEE Trans. Med. Imaging 2010):*
*Heuristic* combination of linearization and localized potentials for fdEIT.

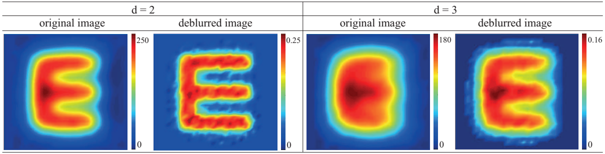


# Planar EIT

*Ts/Lee/Seo/***H.***/Kim (submitted):* Partial inversion plus linearization

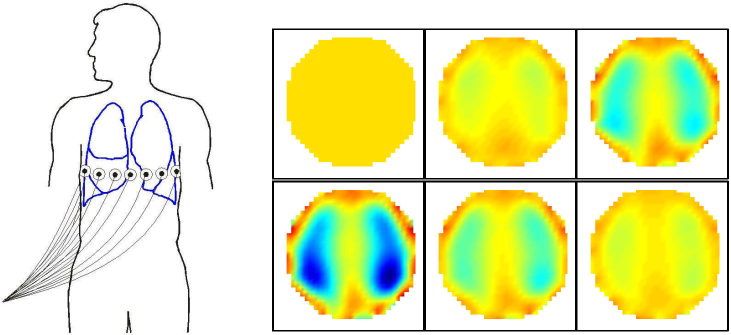


# Lung monitoring - real human data

*BMBF-project on regularizing EIT in the medical and geophysical sciences.*



# Conclusion

- Novel tomography techniques lead to mathematical parameter identification problems.
- Uniqueness questions may have non-trivial answers.

  In diffuse optical tomography:
  - Effective absorption plus jumps in diffusion coefficient (and its derivative) can be reconstructed simultaneously.

  $\rightsquigarrow$ Uniqueness for piecewise constant coefficients, but not for general smooth coefficients
- Non-linearity is main challenge for stable/convergent algos.
  - Shape information is not affected by linearization errors.
  - Stable shape reconstruction is possible.
- Close interplay between uniqueness arguments and convergent reconstruction algorithms.