Exact shape-reconstruction by one-step linearization in EIT

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Mathematical Model

Forward operator of EIT:

$$\Lambda : \ \sigma \mapsto \Lambda(\sigma), \quad \text{"conductivity"} \mapsto \text{"measurements"}$$

- Conductivity: $\sigma \in L^\infty_+(\Omega)$
- Continuum model: $\Lambda(\sigma)$: Neumann-Dirichlet-operator

$$\Lambda(\sigma) : \ g \mapsto u|_{\partial\Omega}, \quad \text{"applied current"} \mapsto \text{"measured voltage"}$$

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega, \quad \sigma \partial_\nu u|_{\partial\Omega} = g \quad \text{on } \partial\Omega. \qquad (1)$$

- Linear elliptic PDE theory:

$$\forall g \in L^2_\diamond(\partial\Omega) \quad \exists! u \in H^1_\diamond(\Omega) \text{ solving (1).}$$

$$\Lambda(\sigma) : L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega) \text{ linear, compact, self-adjoint}$$

Inverse problem

Non-linear forward operator of EIT

$$\Lambda: \ \sigma \mapsto \Lambda(\sigma), \quad L^\infty_+(\Omega) \to \mathcal{L}(L^2_\diamond(\partial\Omega))$$

Inverse problem of EIT:

$$\boxed{\Lambda(\sigma) \mapsto \sigma?}$$

Uniqueness (”Calderón problem“):

- Measurements on complete boundary:
  Calderón (1980), Kohn/Vogelius (1984), Sylvester/Uhlmann (1987), Nachman (1996), Astala/Päivärinta (2006)
- Measurements on part of the boundary:
  *Bukhgeim/Uhlmann (2002), Knudsen (2006), Isakov (2007), Kenig/Sjöstrand/Uhlmann (2007), **H**. (2008), Imanuvilov/Uhlmann/Yamamoto (2009)*

Linearization

Generic approach: Linearization

$$\Lambda(\sigma) - \Lambda(\sigma_0) \approx \Lambda'(\sigma_0)(\sigma - \sigma_0)$$

σ_0: known reference conductivity / initial guess / . . .

$\Lambda'(\sigma_0)$: Fréchet-Derivative / sensitivity matrix.

$$\Lambda'(\sigma_0) : L^\infty_+(\Omega) \to \mathcal{L}(L^2_\diamond(\partial\Omega)).$$

$\rightsquigarrow$ Solve linearized equation for difference $\sigma - \sigma_0$.

Often: $\operatorname{supp}(\sigma - \sigma_0) \subset\subset \Omega$ compact. *(”shape“ / ”inclusion“)*

Linearization

Linear reconstruction method

e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

Solve $\Lambda'(\sigma_0)\kappa \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, then $\kappa \approx \sigma - \sigma_0$.

- Multiple possibilities to measure residual norm and to regularize.
- No rigorous theory for single linearization step.
- Almost no theory for Newton iteration:

  Dobson (1992): (Local) convergence for regularized EIT equation.
  Lechleiter/Rieder(2008): (Local) convergence for discretized setting.
  No (local) convergence theory for non-discretized case!

Linearization

Linear reconstruction method

e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

Solve $\Lambda'(\sigma_0)\kappa \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, then $\kappa \approx \sigma - \sigma_0$.

- Seemingly, no rigorous results possible for single linearization step.
- Seemingly, only justifiable for small $\sigma - \sigma_0$ (local results).

In this talk: Rigorous and global(!) result about the linearization error.

Exact Linearization

Theorem *(**H.**/Seo, accepted to SIAM J. Math. Anal.)*
Let κ, σ, σ_0 piecewise analytic and $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$. Then

(a) $\quad \mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

(b) $\quad \frac{\sigma_0}{\sigma}(\sigma - \sigma_0) \leq \kappa \leq \sigma - \sigma_0$ on the bndry of $\mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

$\mathrm{supp}_{\partial\Omega}$: outer support (= supp, if supp is compact and has conn. complement)

- Exact solution of lin. equation yields correct (outer) shape.
- No assumptions on $\sigma - \sigma_0$!

$\rightsquigarrow$ Single-step linearization error does not affect shape reconstrution.

Proof: Combination of monotony and localized potentials.

Monotony

Monotony *(in the sense of quadr. forms)*:

$$\Lambda'(\sigma_0)(\sigma - \sigma_0) \leq \underbrace{\Lambda(\sigma) - \Lambda(\sigma_0)}_{=\Lambda'(\sigma_0)\kappa} \leq \Lambda'(\sigma_0)\left(\frac{\sigma_0}{\sigma}(\sigma - \sigma_0)\right).$$

Kang/Seo/Sheen (1997), Kirsch (2005), Ide/Isozaki/Nakata/Siltanen/Uhlmann (2007)

Quadratic forms / energy formulation:

$$\int_{\partial\Omega} g\Lambda(\sigma_0)g \,\mathrm{d}s = \int_\Omega \sigma_0 |\nabla u_0|^2 \,\mathrm{d}x$$

$$\int_{\partial\Omega} g\Lambda(\sigma)g \,\mathrm{d}s = \int_\Omega \sigma |\nabla u|^2 \,\mathrm{d}x$$

$$\int_{\partial\Omega} g\left(\Lambda(\sigma_0)'\kappa\right) g \,\mathrm{d}s = -\int_\Omega \kappa |\nabla u_0|^2 \,\mathrm{d}x$$

u_0 (resp. u): solution corresponding to σ_0 (resp. σ) and bndry current g.

Bounds on squares

Exact linearization: $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$

$$\leadsto \boxed{\int_\Omega (\sigma - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x \geq \int_\Omega \kappa |\nabla u_0|^2 \,\mathrm{d}x \geq \int_\Omega \frac{\sigma_0}{\sigma}(\sigma - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x.}$$

for all ”reference solutions“ u_0.

Does this imply

$$\sigma - \sigma_0 \geq \kappa \geq \frac{\sigma_0}{\sigma}(\sigma - \sigma_0) \text{ ?}$$

- Famous concept of inverse problems for PDEs:

  ”Completeness of products“ (of solutions of a PDE)

- Here: *”bounds on squares“* (of gradients of solutions of a PDE).

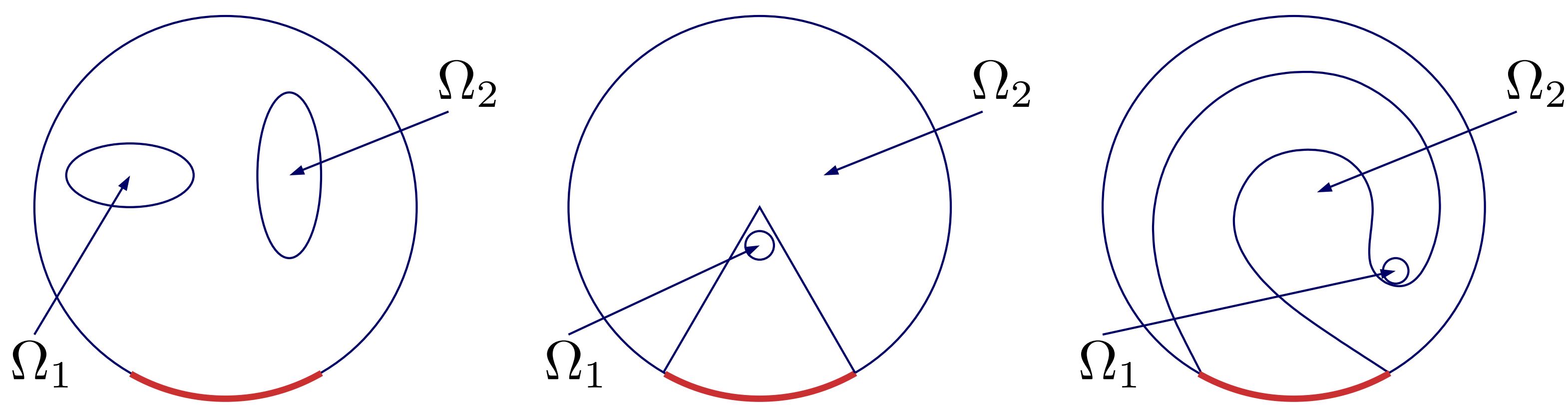
Can we control the ”squares“?

Existence of localized potentials

Theorem*(**H.** 2008)*

If $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$, $B \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ is connected and its boundary contains S, then $\exists$ currents $(g^{(n)})$ s.t. the corresponding reference potentials $(u_0^{(n)})$ fulfill:

$$\int_{\Omega_1} |\nabla u_0^{(n)}|^2 \, \mathrm{d}x \to \infty, \quad \text{and} \quad \int_{\Omega_2} |\nabla u_0^{(n)}|^2 \, \mathrm{d}x \to 0.$$



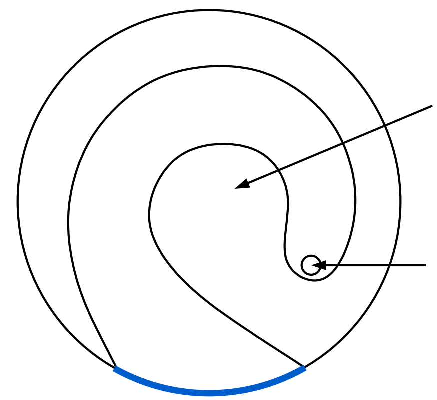
We can make "squares" large on Ω_1 and small on Ω_2.

Bounds on squares

$$\int_\Omega (\sigma - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x \geq \int_\Omega \kappa |\nabla u_0|^2 \,\mathrm{d}x \geq \int_\Omega \frac{\sigma_0}{\sigma}(\sigma - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x.$$

Localized potentials:

Make $|\nabla u_0|^2$ arbitrarily large in a region connected to the boundary but keep it small outside the connecting domain.



$$\mathrm{supp}_{\partial\Omega} \frac{\sigma_0}{\sigma}(\sigma - \sigma_0) = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0) \quad \rightsquigarrow \quad \mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$$

Also: $\frac{\sigma_0}{\sigma}(\sigma - \sigma_0) \leq \kappa \leq \sigma - \sigma_0$ on the bndry of $\mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$

Consequences

Theorem

Let κ, σ, σ_0 piecewise analytic and $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$. Then

(a) $\operatorname{supp}_{\partial\Omega}\kappa = \operatorname{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

(b) $\frac{\sigma_0}{\sigma}(\sigma - \sigma_0) \leq \kappa \leq \sigma - \sigma_0$ on the bndry of $\operatorname{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

Same arguments applied to the Calderón-problem:

$$\Lambda(\sigma) = \Lambda(\sigma_0) \qquad \Longrightarrow \qquad \kappa = 0 :$$

- $\leadsto$ Calderón problem uniquely solvable for piecew. anal. conduct. (already known: *Kohn/Vogelius, 1984*).
- $\leadsto$ Linearized Calderón problem uniquely solvable for p.a. conduct. (already known for piecewise polynomials: *Lechleiter/Rieder, 2008*).

Non-exact Linearization?

Theorem

Let κ, σ, σ_0 piecewise analytic and $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$. Then

(a) $\mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

(b) $\frac{\sigma_0}{\sigma}(\sigma - \sigma_0) \leq \kappa \leq \sigma - \sigma_0$ on the bndry of $\mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

- Existence of exact solution is unknown!
- In practice: finite-dimensional, noisy measurements.

Proof only requires

$$\Lambda'(\sigma_0)(\sigma - \sigma_0) \leq \Lambda'(\sigma_0)\kappa \leq \Lambda'(\sigma_0)\left(\frac{\sigma_0}{\sigma}(\sigma - \sigma_0)\right). \qquad (*)$$

$\rightsquigarrow$ Solve linearized equation s.t. (*) is fulfilled.

Non-exact Linearization

Additional definiteness assumption: $\sigma \geq \sigma_0$.

Assume we are given

- Noisy data $\tilde{\Lambda}_m(\sigma) - \tilde{\Lambda}_m(\sigma_0) \to \Lambda(\sigma) - \Lambda(\sigma_0)$
- Noisy sensitivity $\tilde{\Lambda}'_m(\sigma_0) \to \Lambda'(\sigma_0)$.
- Finite-dim. subspace $V_1 \subset V_2 \subset \ldots \subset L^2_\diamond(\partial\Omega)$ with dense union.

Equip V_k with norm

$$\|g\|^2_{(m)} := \langle (\tilde{\Lambda}_m(\sigma) - \tilde{\Lambda}_m(\sigma_0))g, g \rangle.$$

Minimize (Galerkin approx. of) linearization residual

$$\tilde{\Lambda}(\sigma) - \tilde{\Lambda}(\sigma_0) - \tilde{\Lambda}'(\sigma_0)\kappa_m$$

in the sense of quadratic forms on V_k.

Non-exact Linearization

Theorem *(**H.**/Seo, accepted to SIAM J. Math. Anal.)*
For appropriately chosen $\delta_1, \delta_2 > 0$, every V_k and suff. large m,

$$\exists \kappa_m : \quad -\delta_1 \leq \tilde{\Lambda}(\sigma) - \tilde{\Lambda}(\sigma_0) - \tilde{\Lambda}'(\sigma_0)\kappa_m \leq \delta_2.$$

(in the sense of quadr. forms on V_k, κ_m piecewise analytic)

Every piecewise analytic L^∞-limit κ of a converging subsequence fulfills

(a) $\quad \mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

(b) $\quad \left(\frac{\sigma_0}{\sigma} - \delta_1\right)(\sigma - \sigma_0) \leq \kappa \leq (\delta_2 + 1)(\sigma - \sigma_0)$ on bndry of $\mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

Convergence guaranteed if $\sigma - \sigma_0$ belongs to fin-dim. ansatz space.

$\rightsquigarrow$ *Globally convergent shape reconstruction by one-step linearization.*

Summary and open questions

- The linearization error in EIT does not affect the shape.
- With additional definiteness assumption, we derived a

  local one-step linearization algorithm

  with *globally convergent* shape reconstruction properties.
- Additional definiteness property is typical for shape reconstruction.

Open questions

- Numerical implementation?
- Formulation as Tikhonov regularization with special norms?
- Definiteness only enters in V_k-norm. Can this be replaced by other oszillation-preventing regularization?