Detecting inclusions of mixed type in optical tomography

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Optical tomography

- Optical tomography:
  Use low-energy visible or near infrared light for medical imaging
- General forward model:
  Photon transport models (Boltzmann transport equation)
- For highly scattering media:
  Diffusion approximation for photon density u:

$$\nabla \cdot \sigma \nabla u - \mu u = 0$$

  $\sigma > 0$: diffusion coefficient
  $\mu > 0$: absorption coefficient

- Inverse problem of (diffusive) optical tomography:
  Reconstruct (properties of) σ and μ from pairs of Neumann and Dirichlet boundary values of u.

Mathematical formulation

- $\Omega \subset \mathbb{R}^n$ smoothly bounded domain, $\sigma, \mu \in L^\infty_+(\Omega)$.
- For every input flux $f \in L^2(\partial\Omega)$ there exists a unique solution $u \in H^1(\Omega)$ of

$$\nabla \cdot \sigma \nabla u - \mu u = 0 \quad \text{in } \Omega, \qquad \nu \cdot \sigma \nabla u|_{\partial\Omega} = f \quad \text{on } \partial\Omega,$$

- The Neumann-to-Dirichlet map

$$\Lambda : f \mapsto u|_{\partial\Omega}, \quad L^2(\partial\Omega) \to L^2(\partial\Omega),$$

is linear, compact and self-adjoint and can also be considered as an isomorphism from $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$.

Detecting inclusions

- Consider the special case that

$$\sigma = 1 + \kappa, \quad \mu = 1 + \eta$$

where $\kappa, \eta \in L^\infty(\Omega)$ are compactly supported in Ω.

- Goal:

Determine support of κ and η ("the inclusions") from comparing Λ with reference operator $\Lambda_0 : \ f \mapsto u_0|_{\partial\Omega}$, where

$$\Delta u_0 - u_0 = 0 \quad \text{in } \Omega, \quad \partial_\nu u_0|_{\partial\Omega} = f \quad \text{on } \partial\Omega,$$

i.e., the Neumann-to-Dirichlet operator of a domain without inclusions.

- Fairly recent, fast methods for detecting inclusions:

Linear Sampling / Factorization Method

(LSM: Colton and Kirsch 1996, FM: Kirsch 1998, FM for EIT: Brühl and Hanke 2000)

Virtual measurements

D: smoothly bounded inclusion, i.e. the support of κ and η. $\Omega \setminus \overline{D}$ conn.

- LSM/FM are based on a relation between the (real) measurements Λ and the so-called virtual measurements

$$L: \ \psi \in H^{-1/2}(\partial D) \mapsto v|_{\partial\Omega} \in L^2(\partial\Omega),$$

where v solves $\Delta v - v = 0$ outside D with $\partial_\nu v|_{\partial\Omega} = 0$.

- $\mathcal{R}(L)$ determines D:
  (Traces) of singular solutions Φ_z of

$$\Delta\Phi_z - \Phi_z = \delta_z, \qquad \partial_\nu \Phi_z|_{\partial\Omega} = 0$$

belong to $\mathcal{R}(L)$ if and only if $z \in D$.

If $\mathcal{R}(L)$ is known, then the inclusions can be found by testing for each point $z \in \Omega$ whether $\Phi_z \in \mathcal{R}(L)$ or not.

LSM / FM

Relation between (real) measurements Λ and virtual measurements L:

- LSM: $\mathcal{R}(L) \supseteq \mathcal{R}(\Lambda_0 - \Lambda)$

$$\rightsquigarrow \quad \{z \mid \Phi_z \in R(\Lambda_0 - \Lambda)\} \subseteq D$$

(holds for all kinds of inclusions).

- FM: $\mathcal{R}(L) = \mathcal{R}(|\Lambda_0 - \Lambda|^{1/2})$

$$\rightsquigarrow \quad \left\{z \mid \Phi_z \in R(|\Lambda_0 - \Lambda|^{1/2})\right\} = D$$

(needs additional assumptions on inclusions).

Properties

Properties of LSM/FM:

- Range test

$$\Phi_z \in \mathcal{R}(\Lambda_0 - \Lambda), \text{ resp., } \Phi_z \in \mathcal{R}(|\Lambda_0 - \Lambda|^{1/2})$$

  is easy to implement and extremely fast (no iterations, no forward solutions!)

- FM also yields theoretical uniqueness result.

However,

- LSM is only guaranteed to find a subset of the inclusion.
- FM needs additional assumptions on inclusions.
- Implementation of the range test needs additional threshold parameter. Finding a convergent threshold choosing strategy is still an open problem!

Known results

Assume that either $\kappa, \eta \geq 0$ or $\kappa, \eta \leq 0$.

- Known results for optical tomography:
  - FM works if conductivity jump κ is strictly positive (or negative) and the absorption jump η does not interfere with injectivity of $\Lambda_0 - \Lambda$. *(Hyvönen 2004, Kirsch 2005, G. 2006)*
  - FM works if $\kappa = 0$ and absorption jump η is strictly positive (or negative). *(Hyvönen 2005)*

- Here:
  - Treat combinations of absorption and conductivity perturbations.
  - Treat smooth transitions between inclusions and background *(analogous result for EIT: G. and Hyvönen 2007)*
  - Treat inclusions with unconnected complements.

Concepts of support

$g_1, g_2 \in L^\infty(\Omega)$, $\operatorname{supp} g_1, \operatorname{supp} g_2 \subset \Omega$

Definition ($\partial\Omega$-support):

$\operatorname{supp}_{\partial\Omega} g_1$ is the complement of the set of all $x \in \Omega$ for which there exists a (relatively) open connected $U \subset \overline{\Omega}$ with $x \in U$, $\partial\Omega \cap U \neq \emptyset$, $g_1|_U = 0$ a. e.

Combined $\partial\Omega$-support:

$$\operatorname{supp}_{\partial\Omega}(g_1, g_2) := \operatorname{supp}_{\partial\Omega}(|g_1| + |g_2|).$$

(goes back to analogous definition of the infinity-support by Kusiak and Sylvester 2003)

"$\operatorname{supp}_{\partial\Omega}(g_1, g_2)$ is closed and contains the support of g_1 and g_2 together with all the holes that cannot be connected to the boundary without crossing the support."

Concepts of support

$g_1, g_2 \in L^\infty(\Omega)$, $\operatorname{supp} g_1, \operatorname{supp} g_2 \subset \Omega$

Definition shaded set:

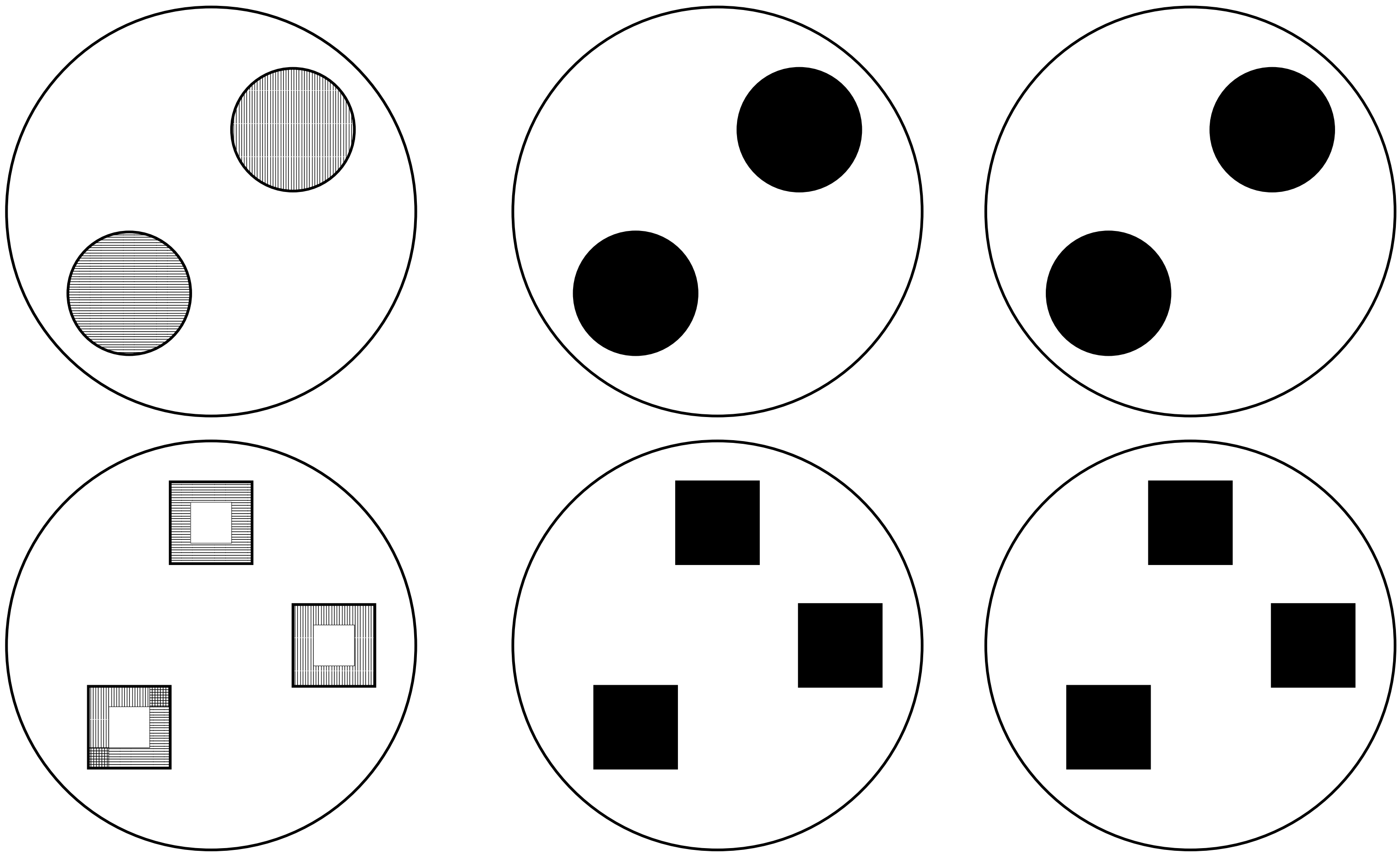
$\mathrm{sh}(g_1, g_2)$ is the set of all $y \in \Omega$ for which there exists a smooth domain $D \subset \Omega$, with $\overline{D} \subset \Omega$ and $\Omega \setminus \overline{D}$ connected, such that $y \in D$ and for each $z \in \partial D$ there exist constants $\epsilon_z, r_z > 0$ such that

$$|g_j| > \epsilon_z I \quad \text{almost everywhere in } B_{r_z}(z),$$

for $j = 1$ or $j = 2$.

"$\mathrm{sh}(g_1, g_2)$ is open and contains $x \in \Omega$ if one cannot travel from x to the boundary $\partial\Omega$ without going over a strictly positive hump in $|g_1|$ or in $|g_2|$."

Examples



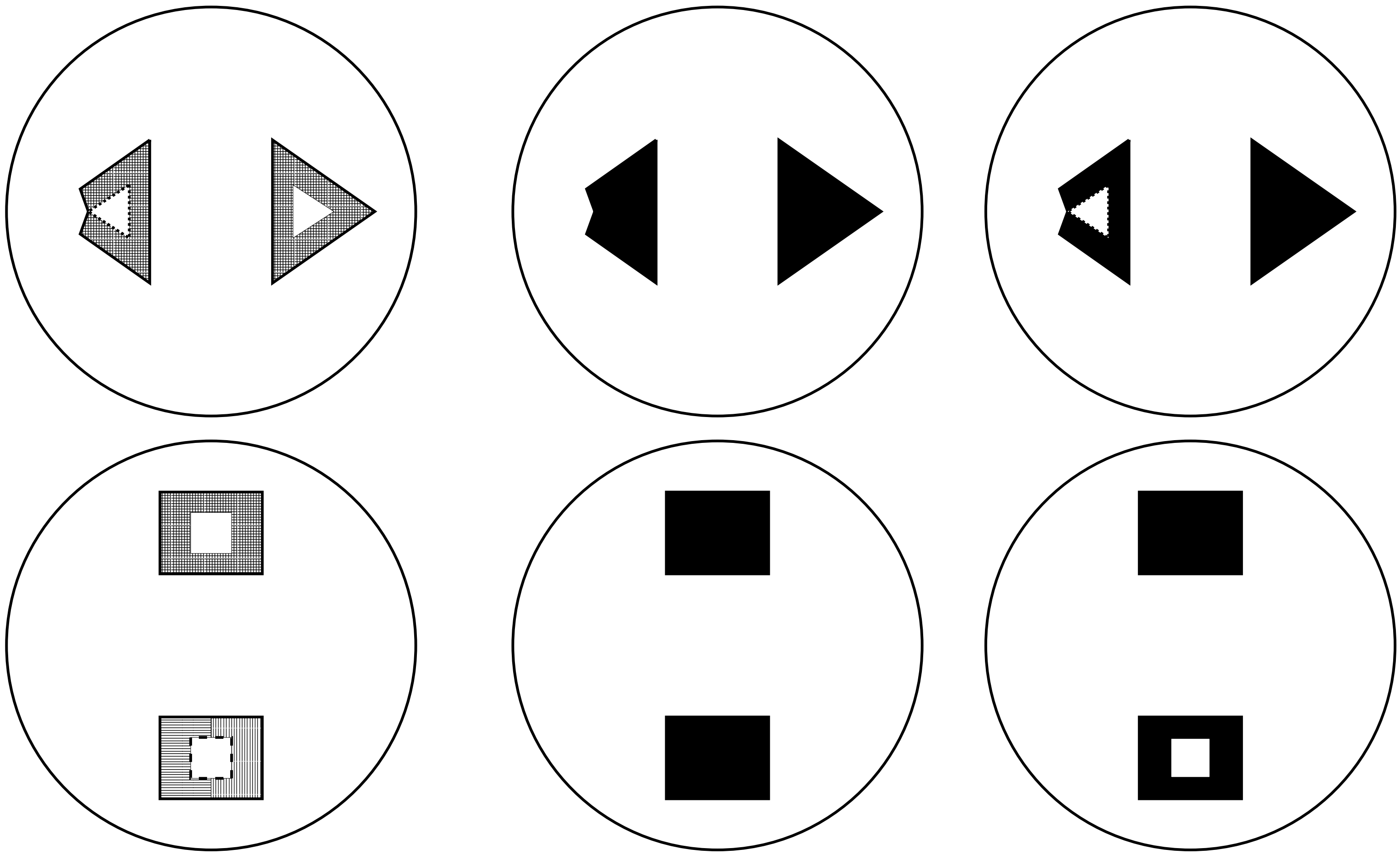
horizontal lines: $\kappa = 1$
vertical lines: $\eta = 1$

$\operatorname{supp}_{\partial\Omega}(\kappa, \eta)$

$\operatorname{sh}(\kappa, \eta)$

Here: $\overline{\operatorname{sh}(\kappa, \eta)} = \operatorname{supp}_{\partial\Omega}(\kappa, \eta)$.

Examples



horizontal lines: $\kappa = 1$

vertical lines: $\eta = 1$

$\mathrm{supp}_{\partial\Omega}(\kappa, \eta)$

$\mathrm{sh}(\kappa, \eta)$

In general: $\overline{\mathrm{sh}(\kappa, \eta)} \subsetneq \mathrm{supp}_{\partial\Omega}(\kappa, \eta)$.

Main result

Theorem

Assume that either $\kappa, \eta \geq 0$ or $\kappa, \eta \leq 0$.

$$\begin{aligned} \Phi_y|_{\partial\Omega} \in \mathcal{R}(|\Lambda_0 - \Lambda|^{1/2}) \quad &\Longrightarrow \quad y \in \operatorname{supp}_{\partial\Omega}(\kappa, \eta) \\ \Phi_y|_{\partial\Omega} \notin \mathcal{R}(|\Lambda_0 - \Lambda|^{1/2}) \quad &\Longrightarrow \quad y \notin \operatorname{sh}(\kappa, \eta) \end{aligned}$$

In other words, the set of points detected by the FM

$$\{y \in \Omega \mid \Phi_y|_{\partial\Omega} \in \mathcal{R}(|\Lambda_0 - \Lambda|^{1/2})\}$$

lies between $\operatorname{sh}(\kappa, \eta)$ and $\operatorname{supp}_{\partial\Omega}(\kappa, \eta)$.

Proof

Main ideas of the proof:

- Monotonicity properties of NtD-Mappings: Let $\sigma_1 \leq \sigma_2 \leq \sigma_3$ and $\mu_1 \leq \mu_2 \leq \mu_3$. Then

$$\mathcal{R}\left\{(\Lambda_1 - \Lambda_2)^{1/2}\right\},\ \mathcal{R}\left\{(\Lambda_2 - \Lambda_3)^{1/2}\right\} \subseteq \mathcal{R}\left\{(\Lambda_1 - \Lambda_3)^{1/2}\right\}.$$

$\rightsquigarrow$ A point is detected by FM if it has a neighbourhood for which FM works. (FM can be applied locally.)

- Every point that is "surrounded by detected points" is itself detected by the FM.

$\rightsquigarrow$ Holes in the inclusion are (falsely) detected by FM.

Remarks

- Result treats ”lower order perturbation” η and ”higher order perturbation” κ in a symmetric way.
- Result needs no regularity or jump conditions for η or κ.
- Result includes known results on FM for optical tomography except for points on the inclusions boundary.
- Result is easily extended to anisotropic diffusivities, non-constant background diffusion or absorption, partial boundary data or other real elliptic equations (e.g., Lamé equations in linear elasticity).

However, perturbations η or κ have to be in the same direction. FM for indefinite problems is still an open question!

Simulated data

Simulated data for the forward problem

- Ω: two-dimensional unit disk
- background diffusion $\sigma_0 = 0.05$, absorption $\mu_0 = 0.5$ corresponding to the optical parameters of a neonatal head
  (cf. Arridge 1999, Hyvönen 2007)
- In inclusions background parameters are doubled.
- Neumann-to-Dirichlet boundary maps $\Lambda_0 - \Lambda$ approximated in trigonometric basis functions using commercial finite element software Comsol.

Implementation of the FM

Let (v_j, λ_j) be the spectral decomposition of $\Lambda_0 - \Lambda$.

Picard criterion:

$$\Phi_y|_{\partial\Omega} \in \mathcal{R}(|\Lambda_0 - \Lambda|^{1/2})$$

if and only if

$$f(y) := \frac{1}{\|\Phi_y|_{\partial\Omega}\|^2_{L^2(\partial\Omega)}} \sum_{j=1}^{\infty} \frac{|\langle \Phi_y|_{\partial\Omega}, v_j \rangle_{L^2(\partial\Omega)}|^2}{|\lambda_j|} < \infty.$$

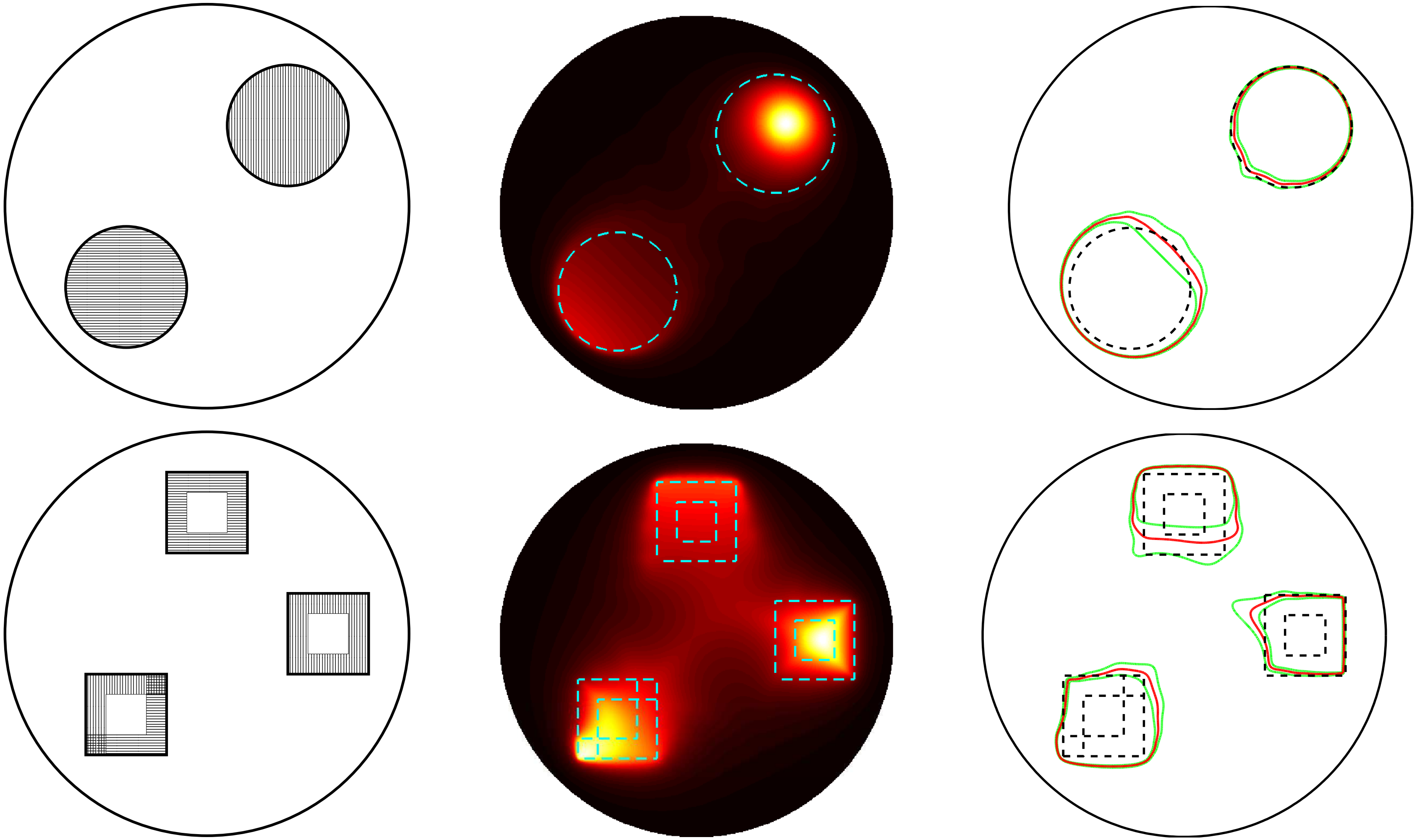
Using a SVD of the finite-dimensional approximation to $\Lambda_0 - \Lambda$ one defines a finite series $\tilde{f}(y) \approx f(y)$.

$$\rightsquigarrow \begin{cases} \tilde{f}(y) \text{ large,} & \text{when } \Phi_y|_{\partial\Omega} \notin \mathcal{R}(|\Lambda_0 - \Lambda|^{1/2}). \\ \tilde{f}(y) \text{ small,} & \text{otherwise.} \end{cases}$$

$\rightsquigarrow$ A plot of (a monotone function of) $\tilde{f}$ should reveal the inclusion.

Convergent treshold choosing strategy for $\tilde{f}$ is still an open problem!

Numerical results



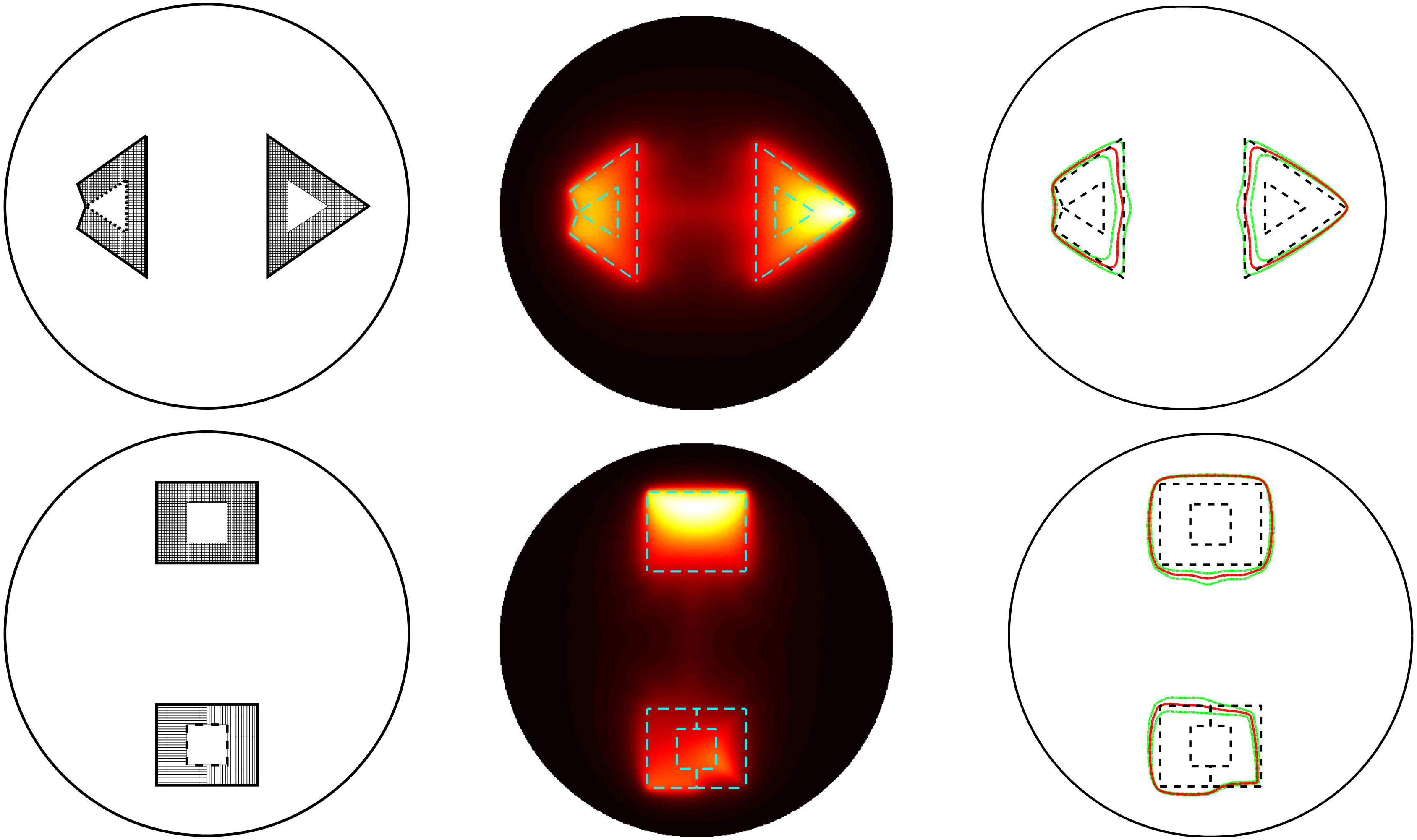
horizontal lines: $\kappa = \sigma_0$

vertical lines: $\eta = \mu_0$

$\tilde{f}(y)$

contour lines of $\tilde{f}$

Numerical results



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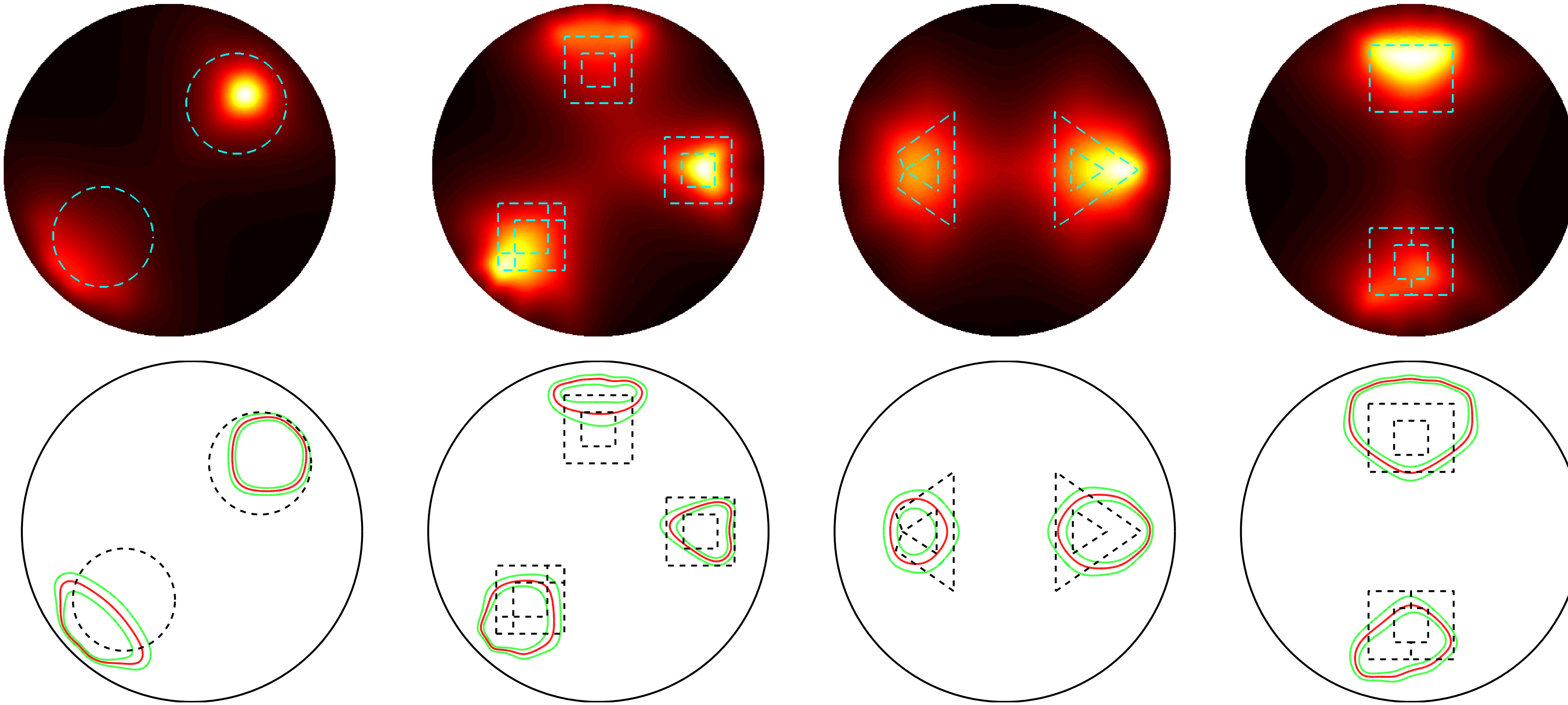
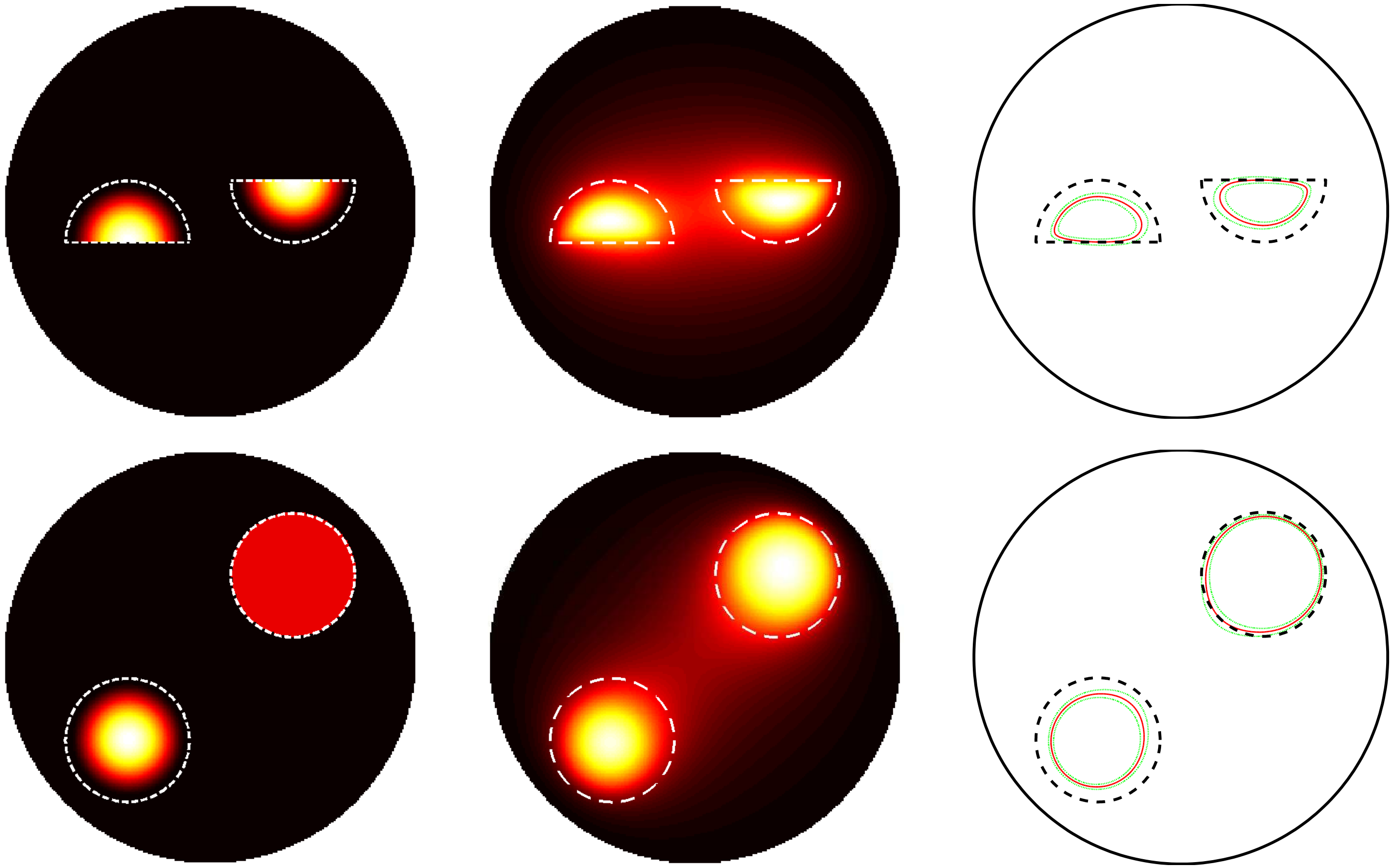
Numerical results



Numerical results with 0.1% noise

Example with smooth transition



real conductivity $\tilde{f}(y)$ contour lines of $\tilde{f}$

Example from EIT with smooth transition *(from G. and Hyvönen 2007)*

Conclusions

- FM simultaneously detects diffusive and absorbing inclusions in optical tomography.
- Holes inside the inclusion are falsely detected by the FM.
- FM detects also smooth deviations, not only jumps.
- Open problems (not only for optical tomography):
  - FM for indefinite problems
  - Convergent treshold choosing strategy for the range test.