Detecting inclusions in complex conductivity EIT without reference measurements

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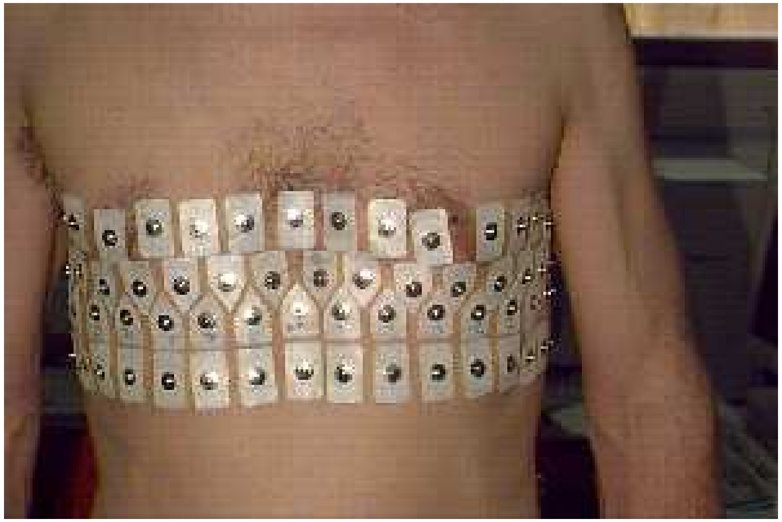
Electrical impedance tomography



(Images taken from EIT group at Oxford Brookes University, published in Wikipedia by William Lionheart)

- Apply one or several input currents to a body and measure the resulting voltages
- Goal: Obtain an image of the interior conductivity distribution.
- Possible advantages:
  - EIT may be less harmful than other tomography techniques,
  - Conductivity contrast is high in many medical applications

Modelling equations

- Time-harmonic Maxwell's equations

$$\begin{aligned} \operatorname{curl} H^\omega - \mathrm{i}\,\omega\epsilon E^\omega &= J + \sigma E^\omega, \\ -\operatorname{curl} E^\omega - \mathrm{i}\,\omega\mu H^\omega &= 0, \end{aligned}$$

ω:	frequency	E^ω:	electric field	ϵ:	dielectricity
J:	appl. currents	H^ω:	magnetic field	μ:	permeability
σ:	conductivity				

- Quasi-static approximation: $\mathrm{i}\,\omega\mu H^\omega \approx 0$

$$\operatorname{curl} H^\omega - \mathrm{i}\,\omega\epsilon E^\omega = J + \sigma E^\omega, \qquad -\operatorname{curl} E^\omega = 0,$$

$\rightsquigarrow$ $E = \nabla u$, with electric potential u:

$\rightsquigarrow$ Conductivity equation

$$\operatorname{div}(\sigma + \mathrm{i}\omega\epsilon)E^\omega = -\operatorname{div} J$$

Electrical impedance tomography

Idealized model for EIT:

- Apply all possible currents g on boundary of the body B
- $\rightsquigarrow$ Electric potential u that solves

$$\nabla \cdot \gamma^\omega \nabla u = 0, \qquad \gamma^\omega \partial_\nu u|_{\partial B} = g$$

($\gamma^\omega = \sigma + \mathrm{i}\omega\epsilon$)

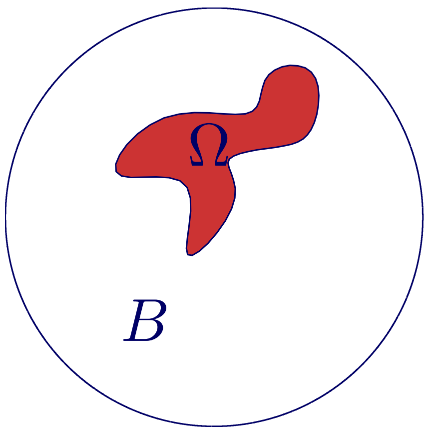
- Measure corresponding potential u everywhere on ∂B.
- $\rightsquigarrow$ Current-to-voltage map $\Lambda : \; g \mapsto u|_{\partial B}$.

Direct Problem: (Standard theory for linear, elliptic PDEs):

- For $\sigma \in L^\infty_+(B)$, $\epsilon \in L^\infty(B)$, $g \in L^2_\diamond(\partial B) := \{\tilde g \in L^2 \; : \; \int \tilde g = 0\}$ there exists a unique solution $u \in H^1_\diamond(B) := H^1(B)/\mathbb{C}$.
- Current-to-voltage map $\Lambda : \; L^2_\diamond(\partial B) \to L^2_\diamond(\partial B)$ is a compact, linear operator.

Detecting inclusions in EIT

Special case of EIT: locate inclusions in known background medium.



Current-to-voltage map with inclusion:

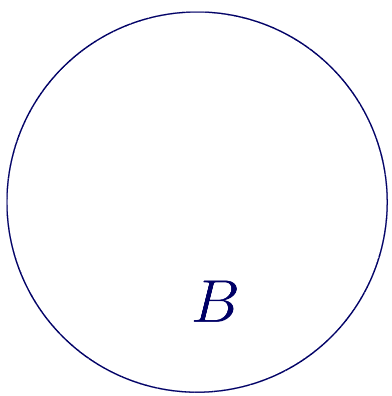
$$\Lambda_1 : \ g \mapsto u_1|_{\partial B},$$

where u_1 solves

$$\nabla \cdot \gamma^\omega \nabla u_1 = 0 \quad \gamma^\omega \partial_\nu u_1|_{\partial B} = g$$

with $\gamma^\omega = \gamma_0^\omega + \gamma_\Omega^\omega \chi_\Omega$,

($\gamma_0^\omega \in \mathbb{C}$: background conductivity).



Current-to-voltage map without inclusion:

$$\Lambda_0 : \ g \mapsto u_0|_{\partial B},$$

where u_0 solves analogous equation with $\gamma^\omega = \gamma_0^\omega$.

"Reference measurements"

Goal: Locate Ω from comparing Λ_1 with Λ_0.

Detecting inclusions

Goal: Locate Ω from comparing Λ_1 with Λ_0.

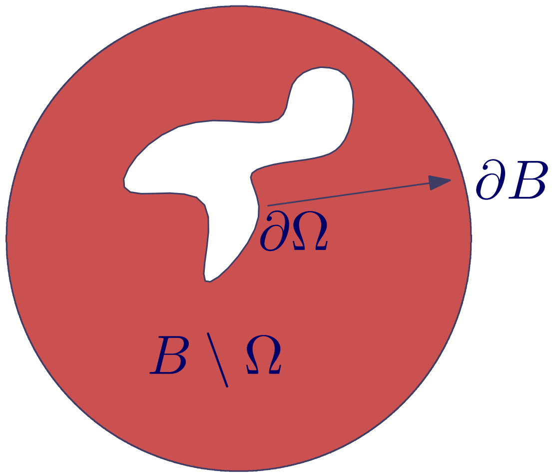
Promising approach: Factorization Method (Kirsch, 1998)

- Non-iterative (no forward solutions of the conductivity equation)
- Yields pointwise, binary criterion whether $z \in \Omega$ or not
  $\rightsquigarrow$ Detect Ω by checking this criterion for every z below S
  ("sampling/probing")
- Needs no a priori information about number and type of inclusions.
- Also implies theoretical uniqueness results.
- Based on functions Φ_z with singularity in sampling point z:

$$\nabla_x \cdot \gamma_0^\omega \nabla_x \Phi_z(x) = d \cdot \nabla_x \delta_z(x) \quad \gamma_0^\omega \partial_{\nu(x)} \Phi_z|_{\partial B} = 0$$

(electric potential of dipole in point z with arbitrary direction d).

Virtual Measurements



ψ: given boundary flux on $\partial\Omega$

$$L : \psi \mapsto v|_{\partial B}, \quad \text{where}$$

$$\nabla \cdot \gamma_0^\omega \nabla v = 0 \quad \text{in } B \setminus \overline{\Omega}, \qquad (1)$$

$$\gamma_0^\omega \partial_\nu v|_{\partial B} = 0 \quad \text{on } \partial B, \qquad (2)$$

$$\gamma_0^\omega \partial_\nu v|_{\partial \Omega} = \psi \quad \text{on } \partial\Omega. \qquad (3)$$

$\mathcal{R}(L)$ determines Ω:

$$\Phi_z|_{\partial B} \in \mathcal{R}(L) \quad \text{if and only if} \quad z \in \Omega$$

Proof:

- If $z \in \Omega$, then the dipole function $\Phi_z|_{B\setminus\overline{\Omega}}$ solves (1)–(3).
- If $\Phi_z|_{\partial B} \in \mathcal{R}(L)$, then Φ_z has no singularity outside Ω due to unique continuation principle.

Factorization Method

$$\Phi_z|_{\partial B} \in \mathcal{R}(L) \quad \text{if and only if} \quad z \in \Omega$$

$\rightsquigarrow$ $\mathcal{R}(L)$ determines Ω.

Key identity of the Factorization Method:

$$\mathcal{R}(L) = \mathcal{R}(|\Lambda_0 - \Lambda_1|^{1/2}).$$

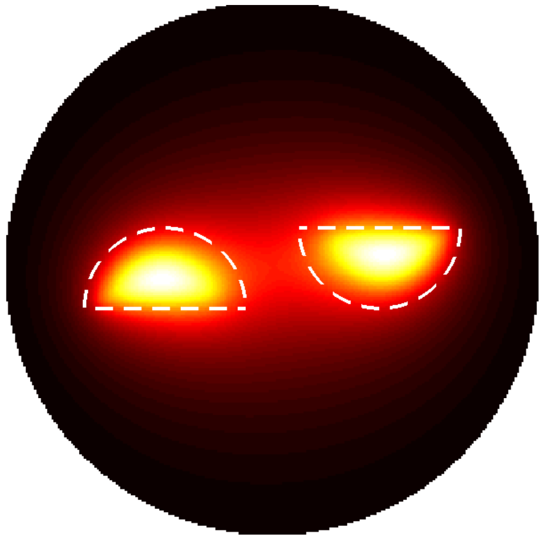
$\rightsquigarrow$ $\mathcal{R}(L)$ (and thus Ω) can be computed from the measurements.

Numerical implementation:

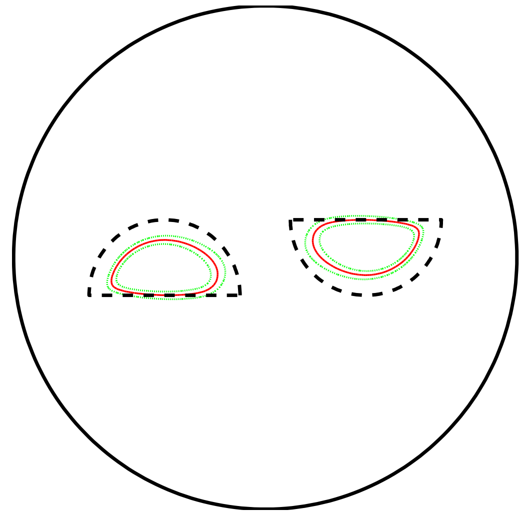
- Calculate regularized approximation of preimage $|\Lambda_0 - \Lambda_1|^{-1/2}\Phi_z|_{\partial B}$
- Plot norm of this approximation as function of z. ("Indicator function")
  - Larger norm $\rightsquigarrow$ preimage does not exist $\rightsquigarrow$ $z \notin \Omega$.
  - Smaller norm $\rightsquigarrow$ preimage exists $\rightsquigarrow$ $z \in \Omega$.

Typical reconstruction

Typical reconstruction for FM:



indicator function



level sets

- Background: real, constant conductivity $\gamma_0^\omega = 1$
- Inclusions: two demicircles,
  real conductivity raises from 1 on circular parts to 2 on line segment
  (smooth deviation at circular part, jump on lines).

History and known results

FM relies on range identity like $\mathcal{R}(L) = \mathcal{R}(|\Lambda_0 - \Lambda_1|^{1/2})$.

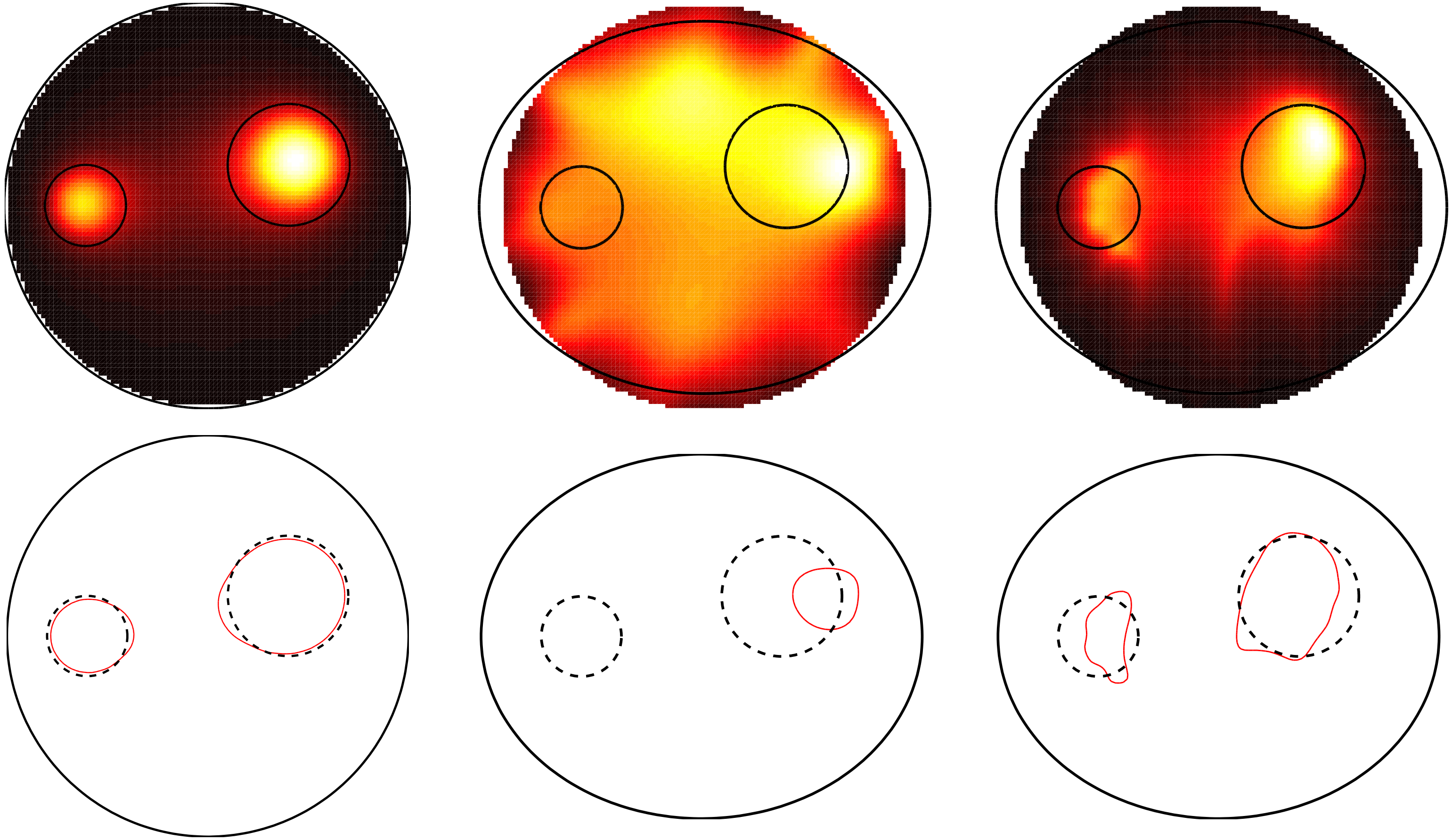
- originally developed by Kirsch, 1998 for inverse scattering problems,
- generalized to real conductivity EIT with inclusions ”with sharp jumps” and connected complement *(Brühl and Hanke, 1999)*,
- extended to electrode models *(Hyvönen, Hakula, Pursiainen, Lechleiter)*,
- detects inclusions with complex conductivity in real conductivity background *(Kirsch, 2005)*,
- detects inclusions ”without sharp jumps” *(G. and Hyvönen 2007)*, ”fills up holes” in case of disconn. complements *(G. and Hyvönen 2008)*.

Here: Variant of FM that detects inclusion in complex conductivity case and works without reference measurements.

Reference measurements

- Factorization method uses difference $\Lambda_1 - \Lambda_0$ between
  - actual measurements Λ_1
  - reference measurements Λ_0 at an inclusion-free body
- Advantage: If reference measurements are available then systematic errors cancel out, e.g., forward modeling errors about the body geometry.
- Disadvantage: If reference measurements have to be simulated (or calculated analytically) then forward modeling errors have a large impact on the reconstructions.
- In medical application, reference measurements at an inclusion-free body are usually not available.

Example



Λ_1: circle	Λ_1: ellipse	Λ_1: ellipse
Λ_0: circle	Λ_0: circle	Λ_0: ellipse
FM for circle	FM for circle	FM for circle

Possible solution

Replace reference measurements by meas. at another frequency:

- Given two frequ. $\omega, \tau > 0$, conductivities $\gamma^\omega, \gamma^\tau$ and NtDs Λ^ω, Λ^τ
- Assume that for all x outside the inclusion Ω

$$\gamma^\omega(x) = \gamma_0^\omega \in \mathbb{C} \quad \text{and} \quad \gamma^\tau(x) = \gamma_0^\tau \in \mathbb{C}$$

- Using $\gamma_0^\omega \Lambda_\omega$ and $\gamma_0^\tau \Lambda_\tau$ scales down conductivity outside Ω to 1.

$\leadsto$ Difference $\gamma_0^\omega \Lambda_\omega - \gamma_0^\tau \Lambda_\tau$ should have similar properties to $\Lambda_1 - \Lambda_0$.

FM should also work with $\gamma_0^\omega \Lambda_\omega - \gamma_0^\tau \Lambda_\tau$ instead of $\Lambda_1 - \Lambda_0$.

- For non-zero frequencies, $\gamma_0^\omega \Lambda_\omega$ is not self-adjoint, so we will have to use its real or imaginary part

$$\Im(A) := \frac{1}{2\mathrm{i}}(A - A^*), \qquad \Re(A) := \frac{1}{2}(A + A^*)$$

for an operator $A: \ L^2_\diamond(S) \to L^2_\diamond(S)$.

fdEIT

Theorem *(G, Seo 2008)*

Let Ω have a connected complement,

$$\gamma^\omega(x) = \gamma_0^\omega + \gamma_\Omega^\omega(x)\chi_\Omega(x), \quad \text{and} \quad \gamma^\tau(x) = \gamma_0^\tau + \gamma_\Omega^\tau(x)\chi_\Omega(x).$$

- If $\Im\left(\frac{\gamma_\Omega^\omega}{\gamma_0^\omega}\right) \in L_+^\infty(\Omega)$ or $-\Im\left(\frac{\gamma_\Omega^\omega}{\gamma_0^\omega}\right) \in L_+^\infty(\Omega)$, then

$$z \in \Omega \quad \text{if and only if} \quad \Phi_z|_{\partial B} \in \mathcal{R}\left(\left|\Im\left(\sigma_0^\omega \Lambda_\omega\right)\right|^{1/2}\right),$$

- If $\Re\left(\frac{\sigma_\Omega^\tau}{\sigma_0^\tau}\right) - \Re\left(\frac{\sigma_\Omega^\omega}{\sigma_0^\omega}\right) - \frac{\Im\left(\frac{\sigma_\Omega^\omega}{\sigma_0^\omega}\right)^2}{\Re\left(\frac{\sigma_\Omega^\omega}{\sigma_0^\omega}\right)} \in L_+^\infty(\Omega)$, then

$$z \in \Omega \quad \text{if and only if} \quad \Phi_z|_{\partial B} \in \mathcal{R}\left(\left|\Re\left(\sigma_0^\omega \Lambda_\omega - \sigma_0^\tau \Lambda_\tau\right)\right|^{1/2}\right).$$

($\tau = 0$ possible and same assertion also holds with interchanged ω and τ).

FM can be used on single non-zero frequency data or on frequency-difference data.

Interpretation

$$z \in \Omega \quad \text{if and only if} \quad \Phi_z|_{\partial B} \in \mathcal{R}\left(\left| \Im\left(\sigma_0^\omega \Lambda_\omega\right)\right|^{1/2} \right),$$

- Using $\Im(\ldots)$ compares Λ_ω to its own adjoint (”phase information”).

⇝ *Inclusion can be found from measurements at a single non-zero frequency without any reference measurements.*

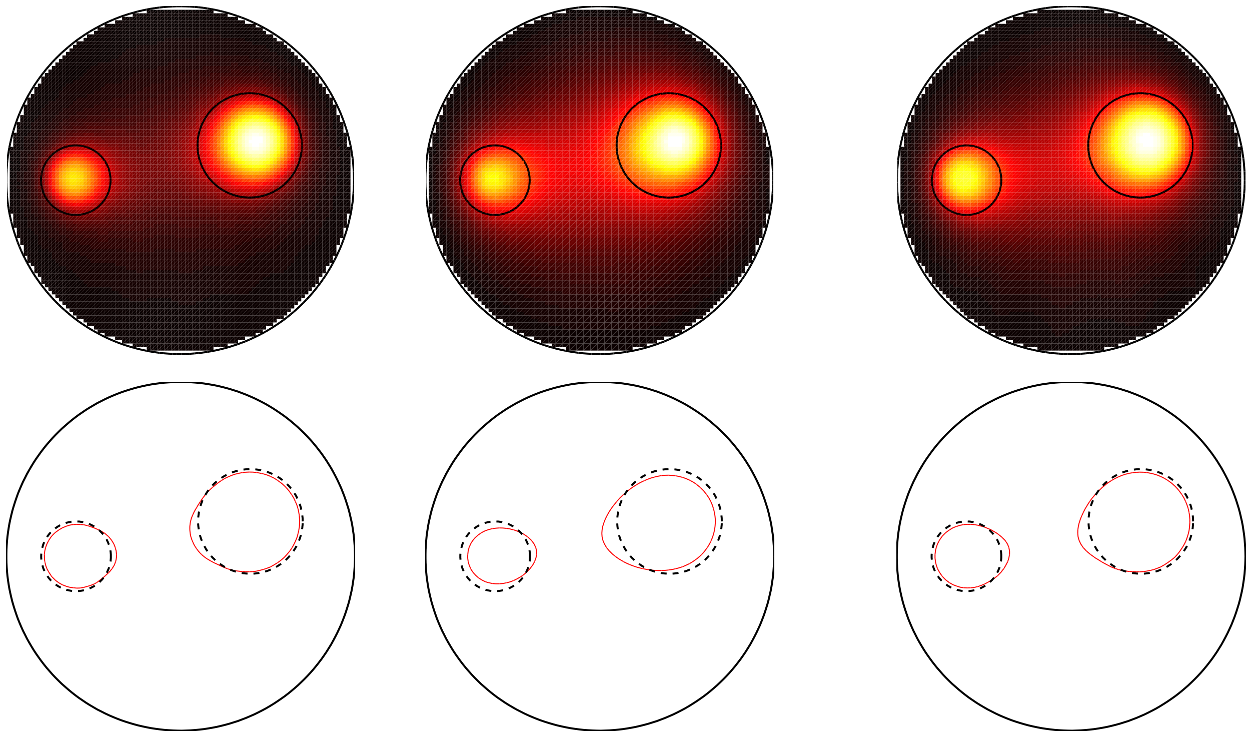
$$z \in \Omega \quad \text{if and only if} \quad \Phi_z|_{\partial B} \in \mathcal{R}\left(\left| \Re\left(\sigma_0^\omega \Lambda_\omega - \sigma_0^\tau \Lambda_\tau\right)\right|^{1/2} \right).$$

- $\Re(\ldots)$ compares two different frequencies.

⇝ *Reference measurements can be replaced by measurements at a different frequency, e.g. by comparing static with non-zero frequency measurements.*

$\omega = 0$ (”static measurements”):	freq. < 1khz
$\omega > 0$, but still low frequency:	1khz < freq. < 500khz

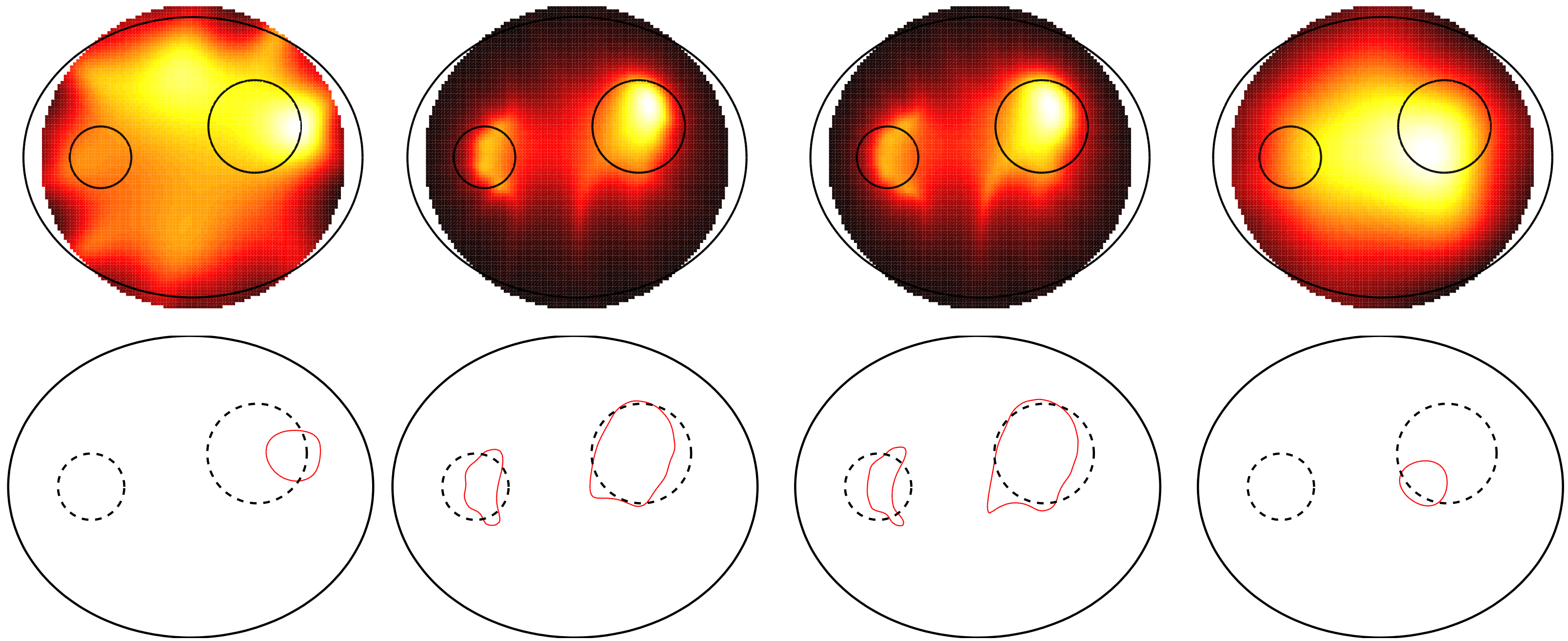
Numerical example



FM with $|\Lambda_0 - \Lambda_1|^{1/2}$ | FM with $|\Im(\sigma_0^\omega \Lambda_\omega)|^{1/2}$ | FM with $|\Re(\sigma_0^\omega \Lambda_\omega - \Lambda_1)|^{1/2}$

(Conductivities: $\sigma = 0.3 - 0.2\chi_\Omega(x)$, $\sigma_\omega(x) = 0.3 + 0.1\mathrm{i} - 0.2\chi_\Omega(x)$.)

Numerical example



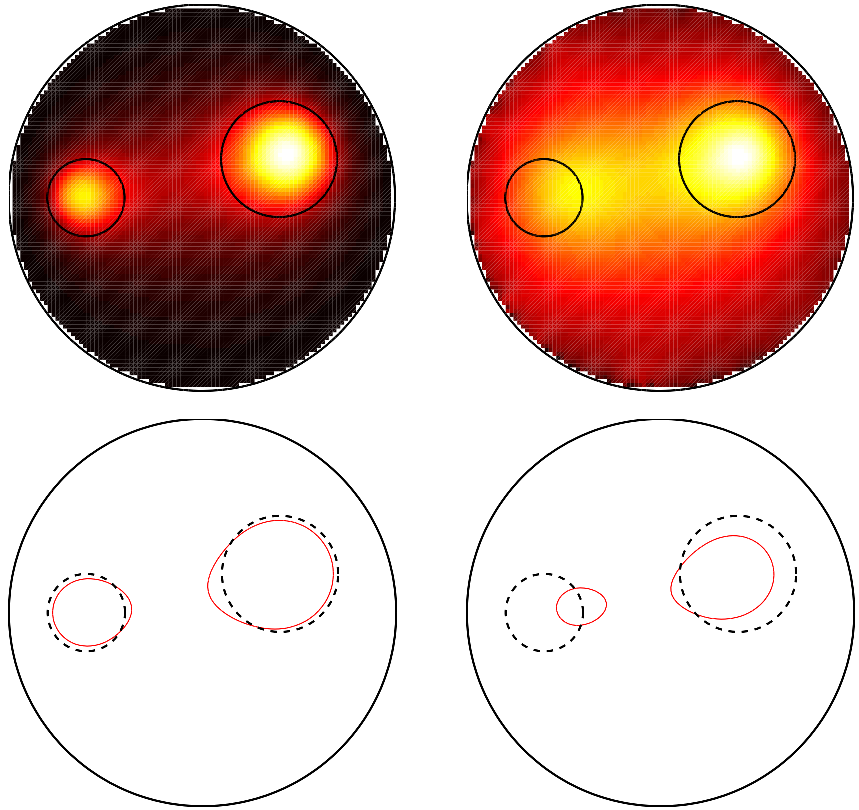
$|\Lambda_0^{\text{circ}} - \Lambda_1|^{\frac{1}{2}}$ $|\Lambda_0^{\text{ellipse}} - \Lambda_1|^{\frac{1}{2}}$ $|\Re\left(\sigma_0^\omega \Lambda_\omega - \Lambda_1\right)|^{\frac{1}{2}}$ $|\Im\left(\sigma_0^\omega \Lambda_\omega\right)|^{\frac{1}{2}}$

Reconstructions of an ellipse-shaped body that is wrongly assumed to be a circle.

Unknown background

- FM for frequency-difference EIT requires no reference data but still needs to know the constant background conductivity
- Heuristic method to estimate this from the data:
  - Eigenvectors for low eigenvalues should belong to highly oscillating potentials that do not penetrate deeply.
  - Most of the quotients of eigenvalues of Λ^ω and Λ^τ should behave like $\gamma_0^\tau / \gamma_0^\omega$.
  - For zero-frequency data $\Lambda^\tau = \Lambda_1$ we use $|\Re(\alpha \Lambda_\omega - \Lambda_1)|^{\frac{1}{2}}$ with the median α of quotients of eigenvalues of Λ^ω and Λ_1.
  - Analogously, the phase of γ_0^ω can be estimated from quotients of real and imaginary part of the eigenvalues of Λ^ω.

Unknown background



no noise $\quad$ 0.1% noise

Reconstructions for unknown background conductivity without and with noise.

Conclusions

Conclusions:

- Simulating reference data makes Factorization Method vulnerable to forward modeling errors.
- Using frequency-difference measurements instead strongly improves FMs robustness. Results are comparable to those with correct reference data.
- Unknown background conductivities can be estimated from the data.

Open problems:

- Scaling the conductivity by simple multiplication only works for constant background conductivity.
- Unsolved problems in the theory of FM: convergent threshold choice, definiteness properties.