Sampling methods for low-frequency electromagnetic imaging

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Low-frequency electromagnetics

Several physically relevant settings for LF-EM imaging.

- Charges/currents applied away from conducting medium

$\rightsquigarrow$ *Inverse scattering problems*

  - Electric charges $\rightsquigarrow$ Electrostatics, Laplace-equation
  - Current loops $\rightsquigarrow$ Magnetostatics, curl-curl-equation
  - Eddy current in conducting objects, parabolic-elliptic equations

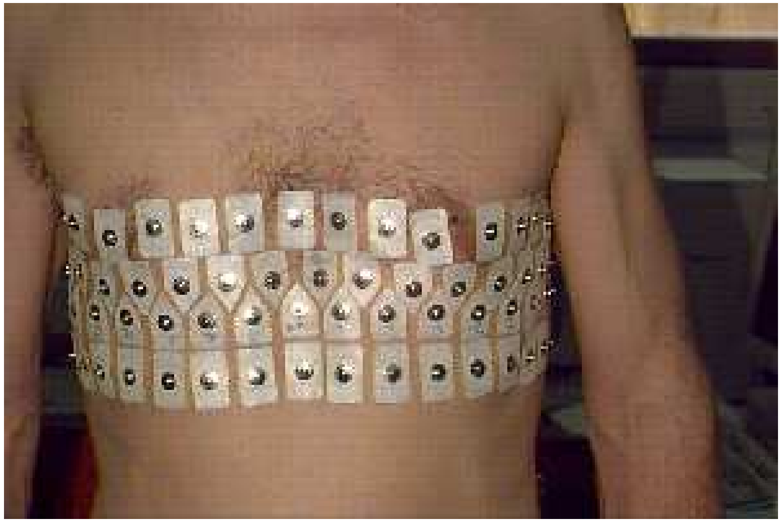
- Currents applied directly to conducting medium

$\rightsquigarrow$ *Electrical Impedance Tomography*

  - freq. $< 1\mathrm{kHz}$ $\rightsquigarrow$ static currents, real conductivity
  - $1\mathrm{kHz} < \mathrm{freq.} < 100\mathrm{kHz}$ $\rightsquigarrow$ phase shifts, complex conductivity

In this talk: EIT with complex conductivity.

Electrical impedance tomography



(Images taken from EIT group at Oxford Brookes University, published in Wikipedia by William Lionheart)

- Apply one or several input currents to a body and measure the resulting voltages
- Goal: Obtain an image of the interior conductivity distribution.
- Possible advantages:
  - EIT may be less harmful than other tomography techniques,
  - Conductivity contrast is high in many medical applications

Electrical impedance tomography

Idealized model for EIT:

- Apply all possible currents g on boundary of the body B

$\rightsquigarrow$ Electric potential u that solves

$$\nabla \cdot \gamma^\omega \nabla u = 0, \qquad \gamma^\omega \partial_\nu u|_{\partial B} = g$$

($\gamma^\omega = \sigma + \mathrm{i}\omega\epsilon$, σ real conductivity, ϵ dielectricity)

- Measure corresponding potential u everywhere on ∂B.

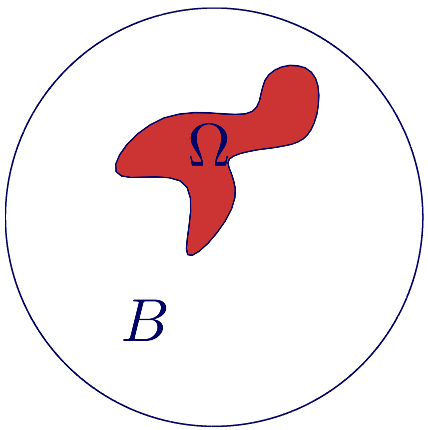
$\rightsquigarrow$ Current-to-voltage map $\Lambda : \quad g \mapsto u|_{\partial B}$.

Direct Problem: (Standard theory for linear, elliptic PDEs):

- For $\sigma \in L^\infty_+(B)$, $\epsilon \in L^\infty(B)$, $g \in L^2_\diamond(\partial B) := \{\tilde{g} \in L^2 \ : \ \int \tilde{g} = 0\}$ there exists a unique solution $u \in H^1_\diamond(B) := H^1(B)/\mathbb{C}$.
- Current-to-voltage map $\Lambda : \ L^2_\diamond(\partial B) \to L^2_\diamond(\partial B)$ is a compact, linear operator.

Detecting inclusions in EIT

Special case of EIT: locate inclusions in known background medium.



Current-to-voltage map with inclusion:

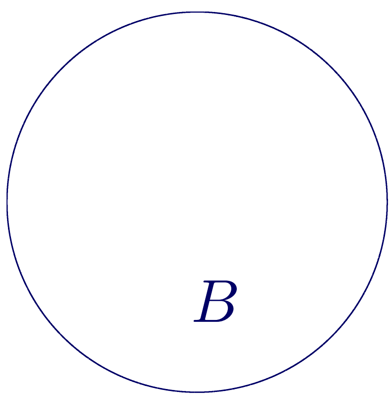
$$\Lambda_1 : \; g \mapsto u_1|_{\partial B},$$

where u_1 solves

$$\nabla \cdot \gamma^\omega \nabla u_1 = 0 \quad \gamma^\omega \partial_\nu u_1|_{\partial B} = g$$

with $\gamma^\omega = \gamma_0^\omega + \gamma_\Omega^\omega \chi_\Omega$,

($\gamma_0^\omega \in \mathbb{C}$: background conductivity).



Current-to-voltage map without inclusion:

$$\Lambda_0 : \; g \mapsto u_0|_{\partial B},$$

where u_0 solves analogous equation with $\gamma^\omega = \gamma_0^\omega$.

"Reference measurements"

Goal: Locate Ω from comparing Λ_1 with Λ_0.

Factorization Method

Factorization Method:

$$\boxed{\Phi_z|_{\partial B} \in \mathcal{R}(|\Lambda_0 - \Lambda_1|^{1/2}) \quad \text{if and only if} \quad z \in \Omega}$$

where

$$\nabla_x \cdot \gamma_0^\omega \nabla_x \Phi_z(x) = d \cdot \nabla_x \delta_z(x), \qquad \gamma_0^\omega \partial_{\nu(x)} \Phi_z|_{\partial B} = 0$$

(electric potential of dipole in point z with arbitrary direction d).

$\rightsquigarrow$ Measurements Λ_1, Λ_0 determine Ω.

Numerical implementation:

- Calculate regularized approximation of preimage $|\Lambda_0 - \Lambda_1|^{-1/2}\Phi_z|_{\partial B}$
- Plot norm of this approximation as function of z. ("Indicator function")
  - Larger norm $\rightsquigarrow$ preimage does not exist $\rightsquigarrow$ $z \notin \Omega$.
  - Smaller norm $\rightsquigarrow$ preimage exists $\rightsquigarrow$ $z \in \Omega$.

History and known results

FM relies on characterization of Ω via $\mathcal{R}(|\Lambda_0 - \Lambda_1|^{1/2})$.

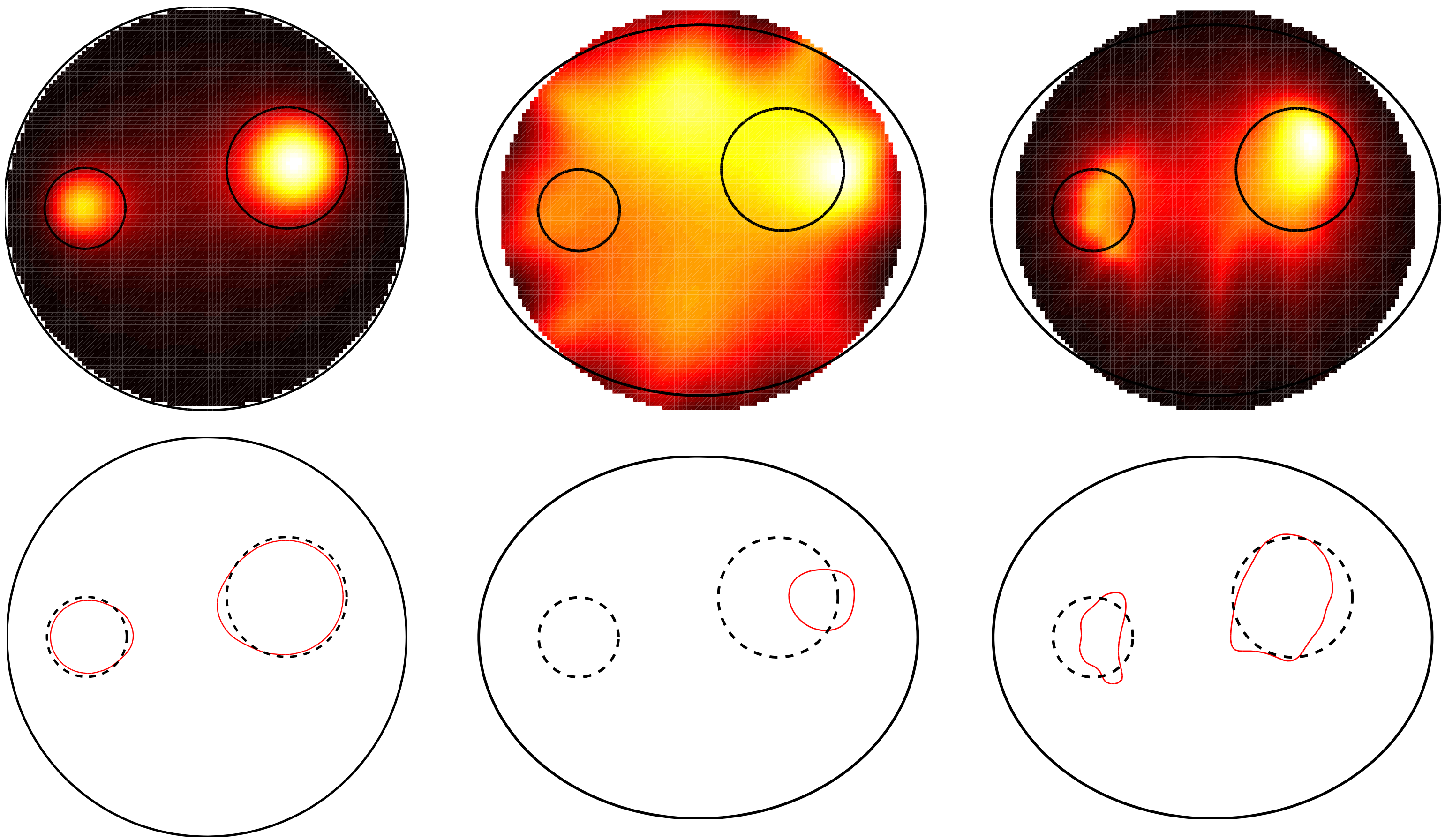
- originally developed by Kirsch, 1998 for inverse scattering problems,
- generalized to real conductivity EIT with inclusions ”with sharp jumps” and connected complement *(Brühl and Hanke, 1999)*,
- extended to electrode models *(Hyvönen, Hakula, Pursiainen, Lechleiter)*,
- detects inclusions with complex conductivity in real conductivity background *(Kirsch, 2005)*,
- detects inclusions ”without sharp jumps” *(G. and Hyvönen 2007)*,
  ”fills up holes” in case of disconn. complements *(G. and Hyvönen 2008)*.

Here: Variant of FM that detects inclusion in complex conductivity case and works without reference measurements.

Reference measurements

- Factorization method uses difference $\Lambda_1 - \Lambda_0$ between
  - actual measurements Λ_1
  - reference measurements Λ_0 at an inclusion-free body
- Advantage: If reference measurements are available then systematic errors cancel out, e.g., forward modeling errors about the body geometry.
- Disadvantage: If reference measurements have to be simulated (or calculated analytically) then forward modeling errors have a large impact on the reconstructions.
- In medical application, reference measurements at an inclusion-free body are usually not available.

Example



Λ_1: circle
Λ_0: circle
FM for circle

Λ_1: ellipse
Λ_0: circle
FM for circle

Λ_1: ellipse
Λ_0: ellipse
FM for circle

Possible solution

Replace reference measurements by meas. at another frequency:

- Given two frequ. $\omega, \tau > 0$, conductivities $\gamma^\omega, \gamma^\tau$ and NtDs Λ^ω, Λ^τ
- Assume that for all x outside the inclusion Ω

$$\gamma^\omega(x) = \gamma_0^\omega \in \mathbb{C} \quad \text{and} \quad \gamma^\tau(x) = \gamma_0^\tau \in \mathbb{C}$$

- Using $\gamma_0^\omega \Lambda_\omega$ and $\gamma_0^\tau \Lambda_\tau$ scales down conductivity outside Ω to 1.

$\rightsquigarrow$ Difference $\gamma_0^\omega \Lambda_\omega - \gamma_0^\tau \Lambda_\tau$ should have similar properties to $\Lambda_1 - \Lambda_0$.

FM should also work with $\gamma_0^\omega \Lambda_\omega - \gamma_0^\tau \Lambda_\tau$ instead of $\Lambda_1 - \Lambda_0$.

- For non-zero frequencies, $\gamma_0^\omega \Lambda_\omega$ is not self-adjoint, so we will have to use its real or imaginary part

$$\Im(A) := \frac{1}{2\mathrm{i}}(A - A^*), \qquad \Re(A) := \frac{1}{2}(A + A^*)$$

for an operator $A: \ L^2_\diamond(S) \to L^2_\diamond(S)$.

fdEIT

Theorem *(G, Seo 2008)*

Let Ω have a connected complement,

$$\gamma^\omega(x) = \gamma_0^\omega + \gamma_\Omega^\omega(x)\chi_\Omega(x), \quad \text{and} \quad \gamma^\tau(x) = \gamma_0^\tau + \gamma_\Omega^\tau(x)\chi_\Omega(x).$$

- If $\Im\left(\frac{\gamma_\Omega^\omega}{\gamma_0^\omega}\right) \in L_+^\infty(\Omega)$ or $-\Im\left(\frac{\gamma_\Omega^\omega}{\gamma_0^\omega}\right) \in L_+^\infty(\Omega)$, then

$$z \in \Omega \quad \text{if and only if} \quad \Phi_z|_{\partial B} \in \mathcal{R}\left(|\Im(\sigma_0^\omega \Lambda_\omega)|^{1/2}\right),$$

- If $\Re\left(\frac{\sigma_\Omega^\tau}{\sigma_0^\tau}\right) - \Re\left(\frac{\sigma_\Omega^\omega}{\sigma_0^\omega}\right) - \frac{\Im\left(\frac{\sigma_\Omega^\omega}{\sigma_0^\omega}\right)^2}{\Re\left(\frac{\sigma_\Omega^\omega}{\sigma_0^\omega}\right)} \in L_+^\infty(\Omega)$, then

$$z \in \Omega \quad \text{if and only if} \quad \Phi_z|_{\partial B} \in \mathcal{R}\left(|\Re(\sigma_0^\omega \Lambda_\omega - \sigma_0^\tau \Lambda_\tau)|^{1/2}\right).$$

($\tau = 0$ possible and same assertion also holds with interchanged ω and τ).

FM can be used on single non-zero frequency data or on frequency-difference data.

Interpretation

$$z \in \Omega \quad \text{if and only if} \quad \Phi_z|_{\partial B} \in \mathcal{R}\left(\left| \Im\left(\sigma_0^\omega \Lambda_\omega\right) \right|^{1/2} \right),$$

- Using $\Im(\ldots)$ compares Λ_ω to its own adjoint (”phase information”).

$\leadsto$ *Inclusion can be found from measurements at a single non-zero frequency without any reference measurements.*

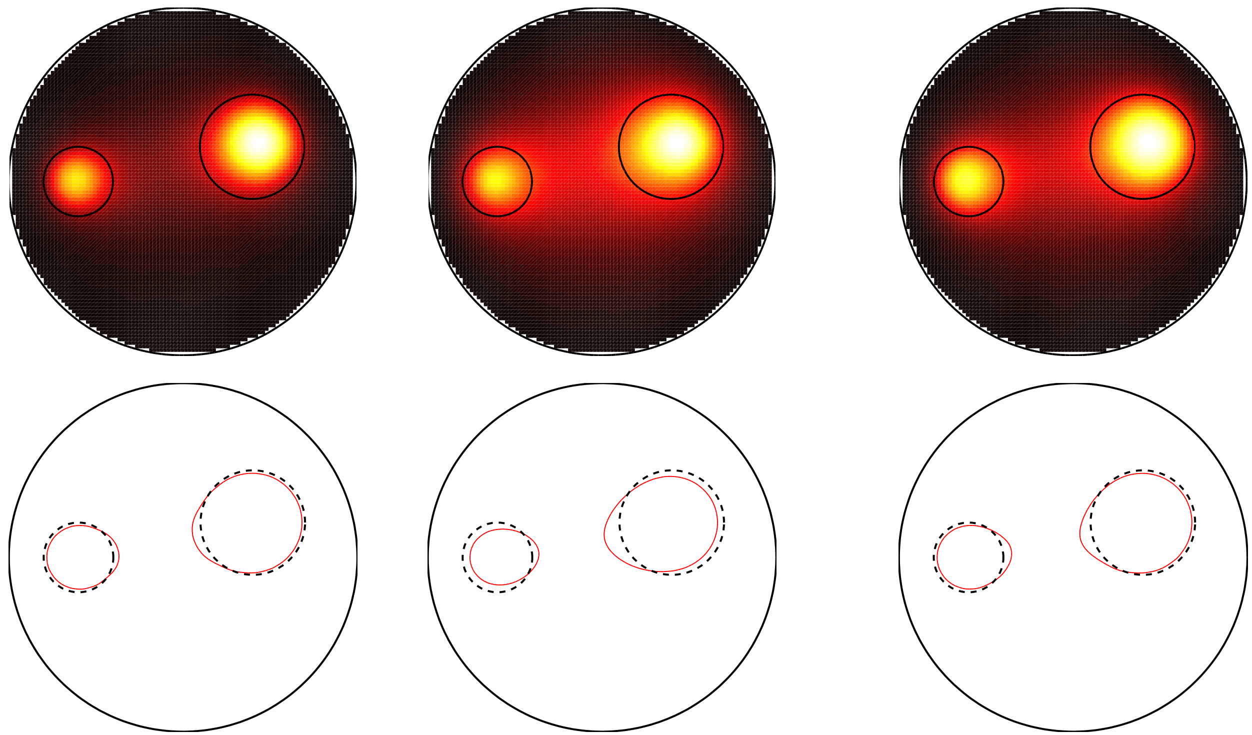
$$z \in \Omega \quad \text{if and only if} \quad \Phi_z|_{\partial B} \in \mathcal{R}\left(\left| \Re\left(\sigma_0^\omega \Lambda_\omega - \sigma_0^\tau \Lambda_\tau\right) \right|^{1/2} \right).$$

- $\Re(\ldots)$ compares two different frequencies.

$\leadsto$ *Reference measurements can be replaced by measurements at a different frequency, e.g. by comparing static with non-zero frequency measurements.*

$\omega = 0$ (”static measurements”):	freq. < 1khz
$\omega > 0$, but still low frequency:	1khz < freq. < 500khz

Numerical example



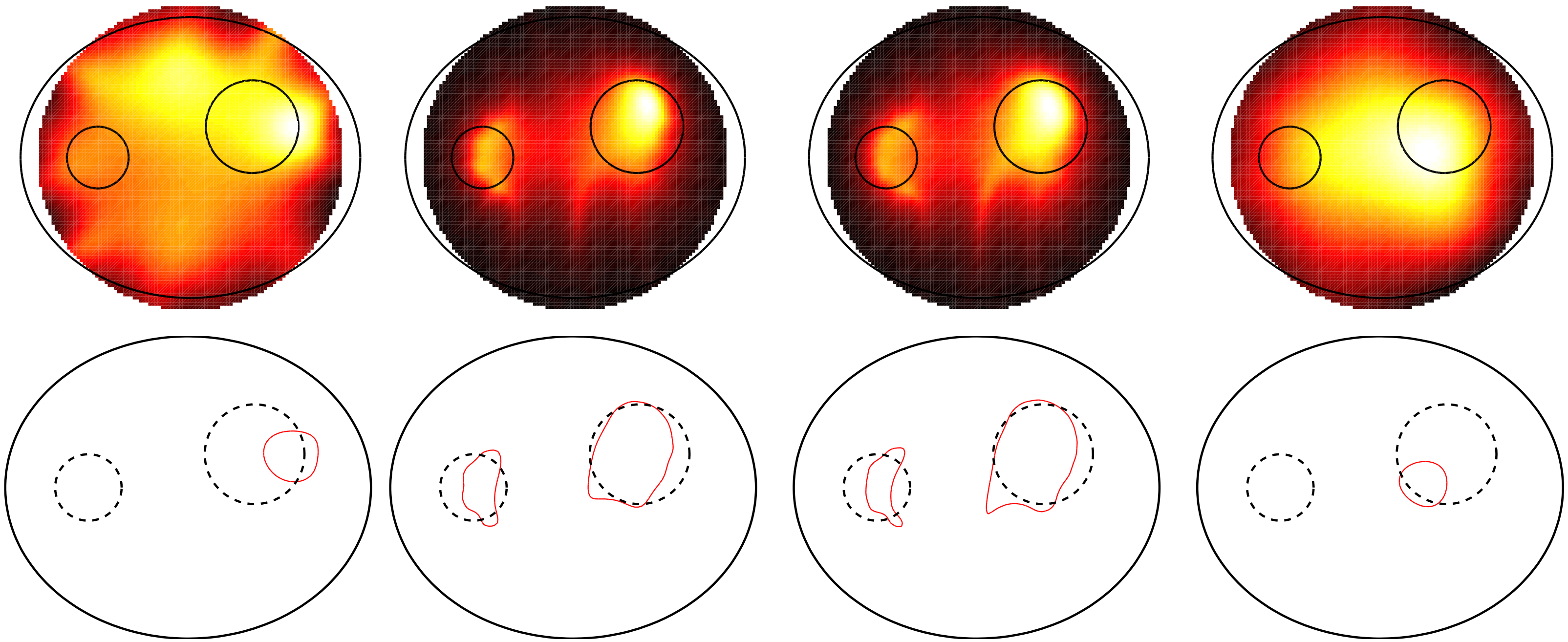
FM with $|\Lambda_0 - \Lambda_1|^{1/2}$

FM with $|\Im\left(\sigma_0^\omega \Lambda_\omega\right)|^{1/2}$

FM with $|\Re\left(\sigma_0^\omega \Lambda_\omega - \Lambda_1\right)|^{1/2}$

(Conductivities: $\sigma = 0.3 - 0.2\chi_\Omega(x), \quad \sigma_\omega(x) = 0.3 + 0.1\mathrm{i} - 0.2\chi_\Omega(x)$.)

Numerical example



$|\Lambda_0^{\mathrm{circ}} - \Lambda_1|^{\frac{1}{2}}$ $|\Lambda_0^{\mathrm{ellipse}} - \Lambda_1|^{\frac{1}{2}}$ $|\Re\left(\sigma_0^{\omega}\Lambda_{\omega} - \Lambda_1\right)|^{\frac{1}{2}}$ $|\Im\left(\sigma_0^{\omega}\Lambda_{\omega}\right)|^{\frac{1}{2}}$

Reconstructions of an ellipse-shaped body that is wrongly assumed to be a circle.

Conclusions

Conclusions:

- Simulating reference data makes Factorization Method vulnerable to forward modeling errors.
- Using frequency-difference measurements instead strongly improves FMs robustness. Results are comparable to those with correct reference data.

Open problems:

- Scaling the conductivity by simple multiplication only works for constant background conductivity.
- Unsolved problems in the theory of FM: convergent threshold choice, definiteness properties.