Monotonicity arguments in electrical impedance tomography

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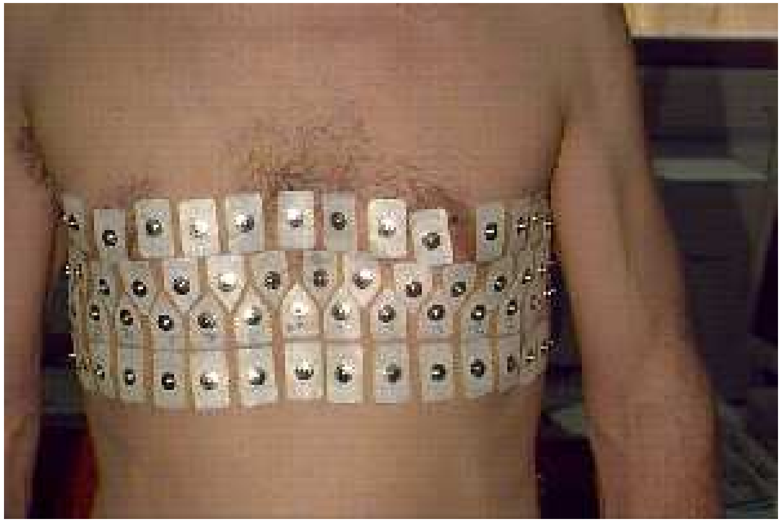
Overview

- Motivation
- Theoretical identifiability results
- The Factorization Method
- Frequency-difference EIT

Motivation

Electrical impedance tomography

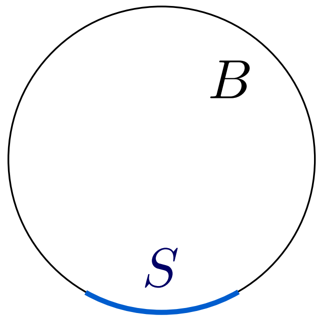


(Images taken from EIT group at Oxford Brookes University, published in Wikipedia by William Lionheart)

- Apply one or several input currents to a body and measure the resulting voltages
- Goal: Obtain an image of the interior conductivity distribution.
- Possible advantages:
  - EIT may be less harmful than other tomography techniques,
  - Conductivity contrast is high in many medical applications

Electrical impedance tomography

Simple mathematical model for EIT:



B:	bounded domain
$S \subseteq \partial B$:	relatively open subset
σ:	electrical conductivity in B
g:	applied current on S

$\rightsquigarrow$ Electric potential u that solves

$$\nabla \cdot \sigma \nabla u = 0, \qquad \sigma \partial_\nu u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else.} \end{cases}$$

Direct Problem: (Standard theory of elliptic PDEs):

For all $\sigma \in L^\infty_+(B)$, $g \in L^2_\diamond(\partial B)$ there exists a unique solution $u \in H^1_\diamond(B)$.

Inverse Problems of EIT:

How can we reconstruct (properties of) σ from measuring $u|_S \in L^2_\diamond(S)$ for one or several input currents g?

Identifiability results

Regularity assumptions on σ

- Most general, "natural" assumption: $\sigma \in L^\infty_+(B)$.
- Slightly less general:
  $\sigma \in L^\infty_+(B)$ and σ satisfies (UCP) in conn. neighborhoods U of S,

$$\nabla \cdot \sigma \nabla u = 0 \text{ and } \begin{cases} u|_S = 0,\ \sigma \partial_\nu u|_S = 0 & \Longrightarrow \quad u = 0. \\ u|_V = \text{const.}, V \subset U \text{ open} & \Longrightarrow \quad u = \text{const.} \end{cases}$$

  For $B \subset \mathbb{R}^2$, (UCP) is satisfied for all $\sigma \in L^\infty_+(B)$.

  For $B \subset \mathbb{R}^n$, $n \geq 3$, (UCP) is satisfied, e.g., for Lipschitz continuous σ.
- For $\sigma \in C^2(\overline{B})$, the EIT equation can be transformed to the Schrödinger equation

$$(\Delta - q)\tilde{u} = 0 \quad \text{with } q = \frac{\Delta\sqrt{\sigma}}{\sqrt{\sigma}}$$

- "Strongest" regularity assumption: σ analytic or piecewise analytic

Identifiability results

Calderon problem with partial data:

Is σ uniquely determined by the (local) current-to-voltage map

$$\Lambda_\sigma : L^2_\diamond(S) \to L^2_\diamond(S), \quad g \mapsto u|_S \text{ ?}$$

For measurements on whole boundary $S = \partial B$:

- Identifiability question posed by Calderon 1980.
- For smooth σ (essentially $\sigma \in C^2$) answered positively by Sylvester and Uhlmann 1987 for $n \geq 3$ and by Nachmann 1996 for $n = 2$.
- For $n = 2$ and general $\sigma \in L^\infty_+$ answered positively by Astala and Päivärinta 2006.
- Still an open question for $\sigma \in L^\infty_+$ (with or without (UCP)) for $n \geq 3$.

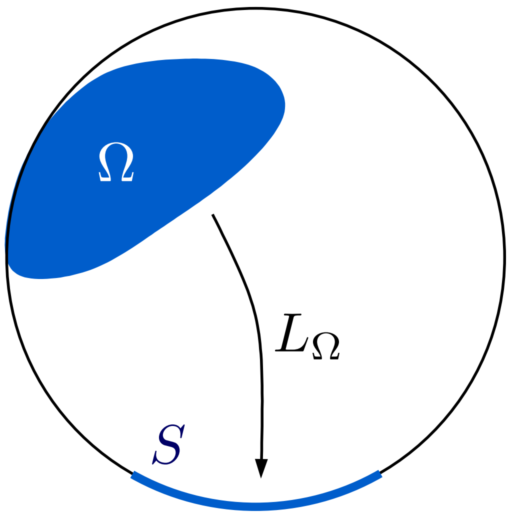
Identifiability results

For measurements on only a part of the boundary $S \subsetneq \partial B$:

- *(Kohn, Vogelius 1984/1985)*:
  Piecewise analytic σ is determined by local voltage-to-current map.
- *(Kenig, Sjöstrand, Uhlmann 2007)*:
  For $n \geq 3$ and with additional condition on boundary parts,
  C^2-conductivities σ are determined by the voltage-to-current map, .
- *(Isakov 2007)*:
  If boundary part is part of a plane or sphere:
  C^2-conductivities σ are determined by the current-to-voltage map, .

Here: A new identifiability result for a large class of L^∞-conductivities (with (UCP)) that is based on comparatively simple monotonicity arguments.

Virtual Measurements



$f \in (H^1_\diamond(\Omega))'$: applied source on Ω

$$L_\Omega : \ (H^1_\diamond(\Omega))' \to L^2_\diamond(S), \quad f \mapsto u|_S,$$

where $u \in H^1_\diamond(B)$ solves

$$\int_B \sigma \nabla u \cdot \nabla v \,\mathrm{d}x = \langle f, v|_\Omega \rangle \quad \text{for all } v \in H^1_\diamond(B).$$

If $\overline{\Omega} \subset B$: $\quad \Delta u = f\chi_\Omega, \quad \sigma \partial_\nu u|_{\partial B} = 0.$

(UCP) yields: If $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$, $B \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ is connected and its boundary contains S then $\mathcal{R}(L_{\Omega_1}) \cap \mathcal{R}(L_{\Omega_2}) = 0$.

Dual operator $L'_\Omega : \ L^2_\diamond(S) \to H^1_\diamond(\Omega), \quad g \mapsto u|_\Omega$, where u solves

$$\nabla \cdot \sigma \nabla u = 0, \qquad \sigma \partial_\nu u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else.} \end{cases}$$

A monotonicity argument

Lemma

Let X, Y be two reflexive Banach spaces, $A \in \mathcal{L}(X, Y)$, $y \in Y$. Then

$$y \in \mathcal{R}(A) \quad \text{iff} \quad |\langle y', y\rangle| \leq C \|A'y'\| \quad \forall y' \in Y'.$$

Corollary

If $\|L'_{\Omega_1} g\| \leq C \|L'_{\Omega_2} g\|$ for all applied currents g, i.e., $\|u|_{\Omega_1}\| \leq C \|u|_{\Omega_2}\|$ for the corresponding potentials u, then $\mathcal{R}(L_{\Omega_1}) \subseteq \mathcal{R}(L_{\Omega_2})$.

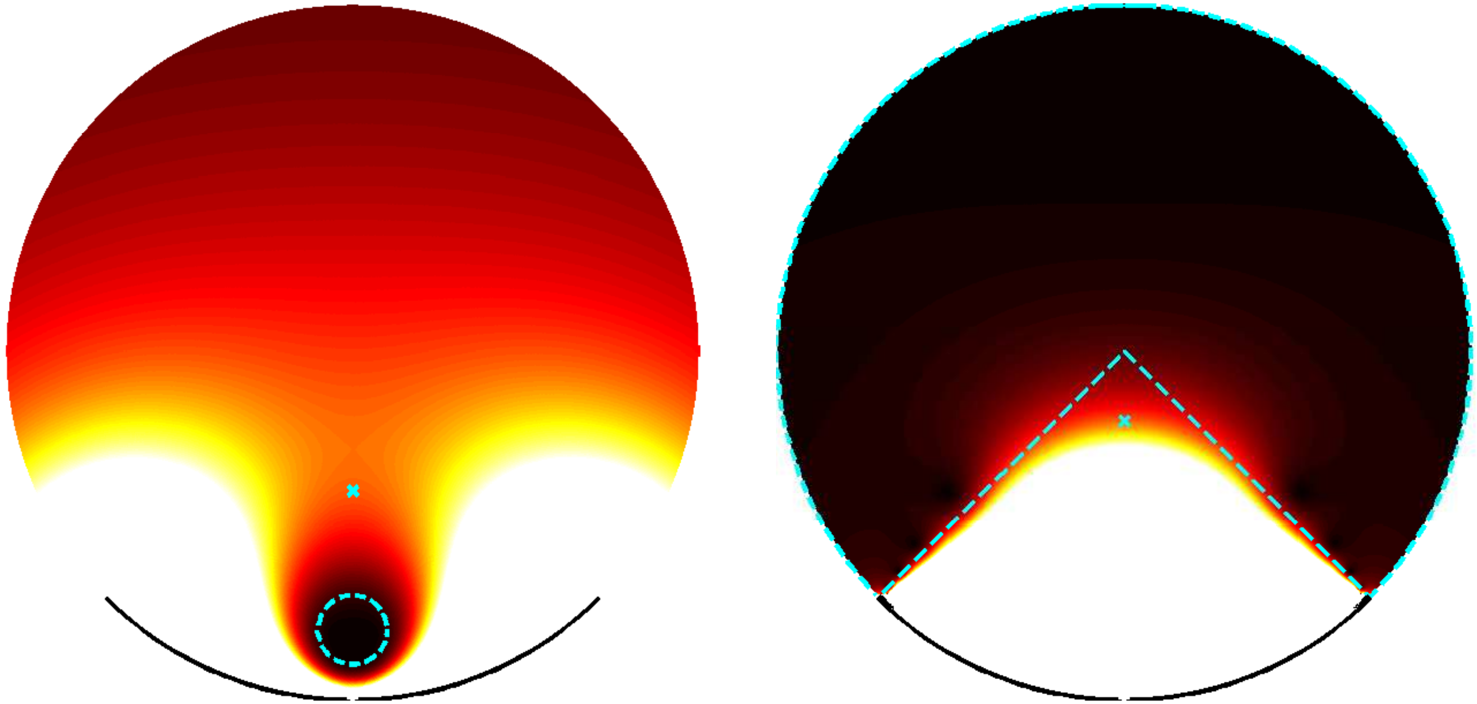
Contraposition

If $\mathcal{R}(L_{\Omega_1}) \not\subseteq \mathcal{R}(L_{\Omega_2})$ then there exist currents (g_n) such that the corresponding potentials (u_n) satisfy

$$\|u_n|_{\Omega_1}\|_{H^1_\diamond(\Omega_1)} \to \infty \quad \text{and} \quad \|u_n|_{\Omega_2}\|_{H^1_\diamond(\Omega_2)} \to 0.$$

"Localized potentials with high energy in Ω_1 and low energy in Ω_2".

Localized potentials



Potentials with high energy around the marked point but low energy in dashed region

Another monotonicity argument

Connection between Calderon problem (with $S \subseteq \partial B$) and localized potentials:

- Monotonicity property:

  Let u_1, u_2 be electric potentials for conductivities σ_1, σ_2 created by the same boundary current $g \in L^2_\diamond(S)$. Then

$$\int_B (\sigma_1 - \sigma_2)|\nabla u_2|^2 \, \mathrm{d}x \geq ((\Lambda_{\sigma_2} - \Lambda_{\sigma_1})g, g) \geq \int_B (\sigma_1 - \sigma_2)|\nabla u_1|^2 \, \mathrm{d}x.$$

$\leadsto$ If $\sigma_1 - \sigma_2 > 0$ in some region where we can localize the electric energy $|\nabla u_1|^2$ then $\Lambda_{\sigma_1} \neq \Lambda_{\sigma_2}$.

"A higher conductivity in such a region can not be balanced out."

A new identifiability result

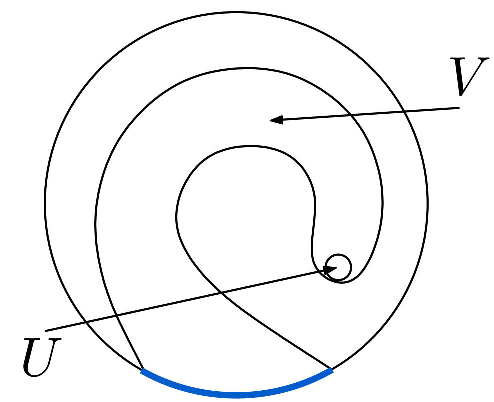
Theorem *(G, 2008)*

Let $\sigma_1, \sigma_2 \in L^\infty_+(B)$ satisfy (UCP) and Λ_{σ_1}, Λ_{σ_2} be the corresponding current-to-voltage-maps.

If $\sigma_2 \geq \sigma_1$ in some neighborhood V of S and $\sigma_2 - \sigma_1 \in L^\infty_+(U)$ for some open $U \subseteq V$ then there exists (g_n) such that

$$\langle(\Lambda_{\sigma_2} - \Lambda_{\sigma_1})g_n, g_n\rangle \to \infty,$$

so in particular $\Lambda_{\sigma_2} \neq \Lambda_{\sigma_1}$.



Two conductivities can be distinguished if one is larger in some part that can be connected to the boundary without crossing a sign change.

Remarks

Remarks

- Theorem covers the Kohn-Vogelius result:
  - $\sigma|_S$ and its derivatives on S are uniquely determined by Λ_σ.
  - Piecewise analytic conductivities σ are uniquely determined.
- Theorem holds for general L^∞_+-conductivities with (UCP).
  (In particular, it is not covered by the recent result of Isakov.)
- Theorem uses only monotonicity properties of real elliptic PDEs, thus also holds e.g. for linear elasticity, electro- and magnetostatics.

However,

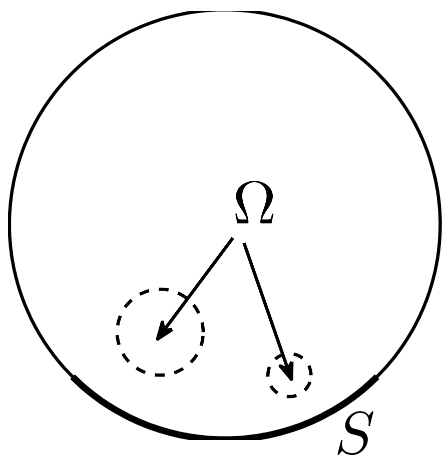
- Theorem needs a neighborhood without sign change. It cannot distinguish infinitely fast oscillating C^∞-conductivities from constant ones.

  The identifiability question for general L^∞_+-conductivities (with or without (UCP)) for $n \geq 3$ or partial boundary data is still open.

The Factorization Method

Detecting inclusions in EIT

Special case of EIT: locate inclusions in known background medium.



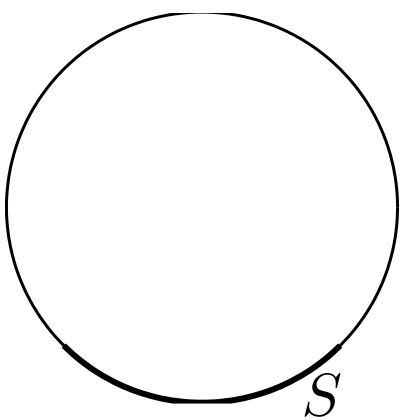
Current-to-voltage map with inclusion:

$$\Lambda_1 : \ g \mapsto u_1|_{\partial B},$$

where u_1 solves

$$\nabla \cdot \sigma \nabla u_1 = 0 \quad \partial_\nu u_1|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else,} \end{cases}$$

with $\sigma = 1 + \sigma_1 \chi_\Omega$, $\sigma_1 \in L^\infty_+(\Omega)$.



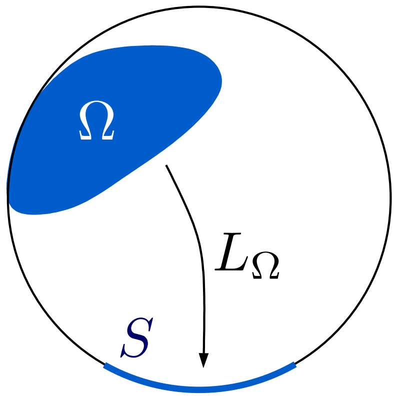
Current-to-voltage map without inclusion:

$$\Lambda_0 : \ g \mapsto u_0|_{\partial B},$$

where u_0 solves the analogous equation with $\sigma = 1$.

Goal: Reconstruct Ω from comparing Λ_1 with Λ_0.

Virtual measurements again



$f \in (H^1_\diamond(\Omega))'$: applied source on Ω

$$L_\Omega : \; (H^1_\diamond(\Omega))' \to L^2_\diamond(S), \quad f \mapsto u|_S,$$

where $u \in H^1_\diamond(B)$ solves (for $\overline{\Omega} \subset B$)

$$\nabla \cdot \sigma \nabla u = f\chi_\Omega, \quad \sigma \partial_\nu u|_{\partial B} = 0.$$

For $B \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ connected with S:

$$\overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset \quad \Longrightarrow \quad \mathcal{R}(L_{\Omega_1}) \cap \mathcal{R}(L_{\Omega_2}) = 0$$

More constructive relation (for $z \notin \partial\Omega$, $B \setminus \overline{\Omega}$ connected with S):

$z \in \Omega$ if and only if $\Phi_z|_S \in \mathcal{R}(L_\Omega)$

with dipole potentials Φ_z, i.e., solutions of

$\Delta \Phi_z = d \cdot \nabla \delta_z$ and $\partial_\nu \Phi_z|_{\partial B} = 0$ (d: arbitrary direction)

$\rightsquigarrow$ $\mathcal{R}(L_\Omega)$ *determines* Ω.

Range characterization

$\mathcal{R}(L_\Omega)$ determines unknown inclusion Ω:

$$z \in \Omega \quad \text{if and only if} \quad \Phi_z|_S \in \mathcal{R}(L_\Omega)$$

In many cases, $\mathcal{R}(L_\Omega)$ can be computed from the measurements Λ_1, Λ_0:

$$\mathcal{R}(L_\Omega) = \mathcal{R}(|\Lambda_0 - \Lambda_1|^{1/2})$$

(Proof uses a factorization of $\Lambda_0 - \Lambda_1$ in L_Ω, L_Ω^* and an auxiliary operator).

Factorization Method (FM):

Locate Ω from measurements Λ_1 and reference measurements Λ_0 by

- computing dipole potentials Φ_z for all points $z \in B$
- find all points z where $\Phi_z \in \mathcal{R}(|\Lambda_0 - \Lambda_1|^{1/2})$,
  e.g., by plotting the norm of regularized approximations to

$$|\Lambda_0 - \Lambda_1|^{-1/2}\Phi_z.$$

History and known results

FM relies on range identity like $\mathcal{R}(L_\Omega) = \mathcal{R}(|\Lambda_0 - \Lambda_1|^{1/2})$.

- FM originally developed by Kirsch for inverse scattering problems and extended to different settings and boundary conditions (with Arens, Grinberg)
- FM generalized to EIT (Brühl/Hanke, 1999)
- FM extended to many applications including electrostatics (Hähner), EIT with electrode models (Hyvönen, Hakula, Pursiainen, Lechleiter), EIT in half-space (Schappel),
  harmonic vector fields (Kress), Stokes equations (Tsiporin),
  optical tomography (Hyvönen, Bal, G), linear elasticity (Kirsch),
  general real elliptic problems (G),
  parabolic-elliptic problems (Frühauf, G, Scherzer)

All these results rely on a parameter jump.
Can FM also detect smooth transitions from background to inclusion?

Monotonicity arguments

Λ_1: NtD for $\sigma = 1 + \sigma_1 \chi_{\Omega_1}$, Λ_2: NtD for $\sigma = 1 + \sigma_2 \chi_{\Omega_2}$, $\sigma_1, \sigma_2 \geq 0$

- Monotony between conductivity and measurements (NtDs):

$$\sigma_1 \chi_{\Omega_1} \leq \sigma_2 \chi_{\Omega_2} \quad \Longrightarrow \quad (\Lambda_1 g, g) \geq (\Lambda_2 g, g) \quad \text{for all } g \in L^2_\diamond(S)$$

- Together with range monotony:

$$\sigma_1 \chi_{\Omega_1} \leq \sigma_2 \chi_{\Omega_2} \quad \Longrightarrow \quad \mathcal{R}((\Lambda_0 - \Lambda_1)^{1/2}) \subseteq \mathcal{R}((\Lambda_0 - \Lambda_2)^{1/2})$$

$\leadsto$ Result of range tests $\Phi_z \in \mathcal{R}((\Lambda_0 - \Lambda_1)^{1/2})$ is monotonous w.r.t. the inclusions size and contrast.

$\leadsto$ FM-theory can be extended to irregular inclusions (e.g. with smooth transitions) by estimating them from above and below by regular inclusions with sharp jumps.

FM with irregular inclusions

Λ_1: NtD for $\sigma = 1 + \sigma_1 \chi_{\Omega_1}$, Λ_0: NtD for $\sigma = 1$.

Theorem *(G, Hyvönen 2007)*

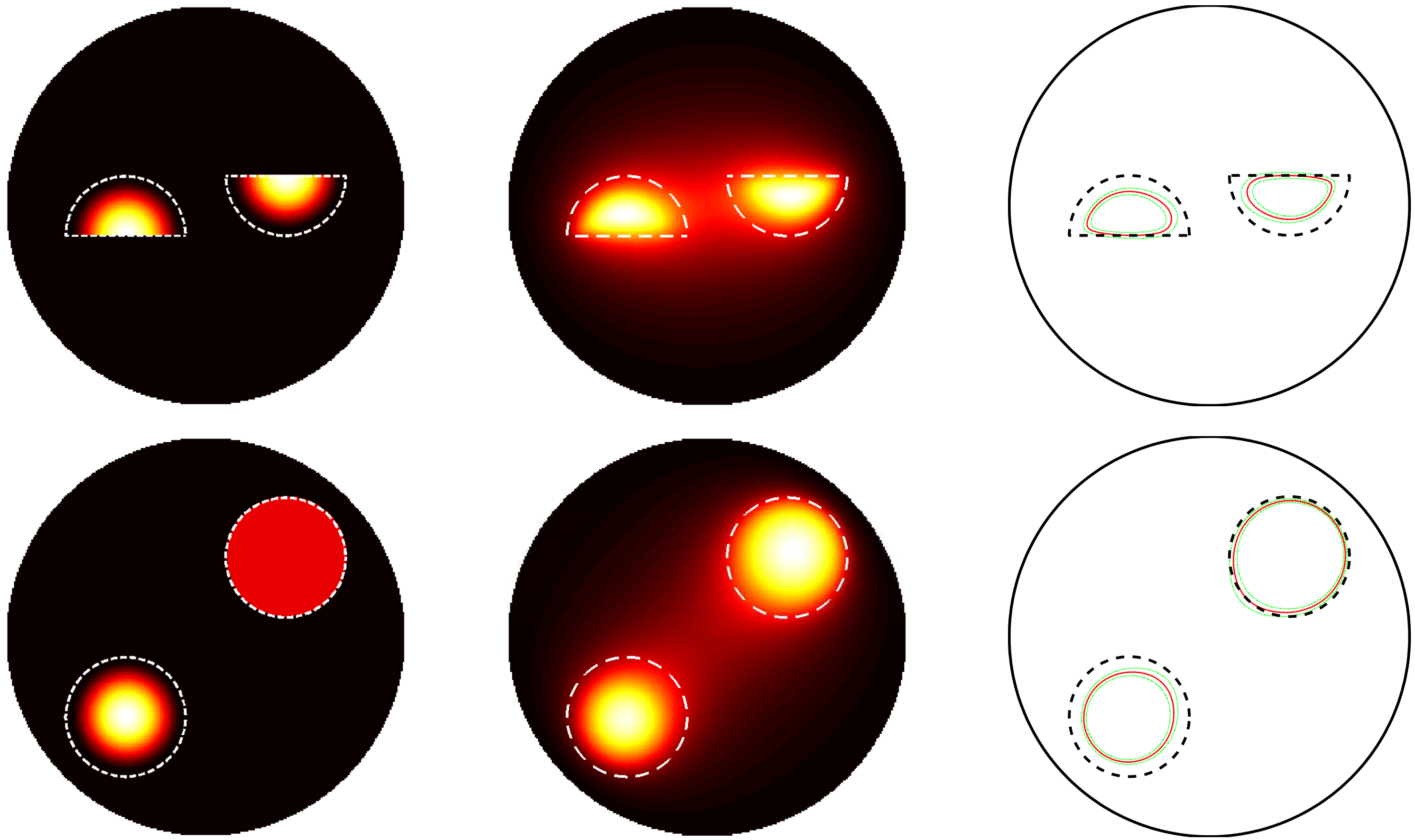
Let $\sigma_1 \geq 0$ and Ω have a connected complement.

- $\Phi_z \in \mathcal{R}((\Lambda_0 - \Lambda_1)^{1/2})$ for every $z \in \Omega$ for which σ_1 is locally in L^∞_+.
- $\Phi_z \notin \mathcal{R}((\Lambda_0 - \Lambda_1)^{1/2})$ for every $z \notin \Omega$.

(and an analogous results holds for $\sigma_1 \leq 0$.)

FM does not only find inclusions where a parameter “jumps”,
but also where it merely “differs” from a known background value.

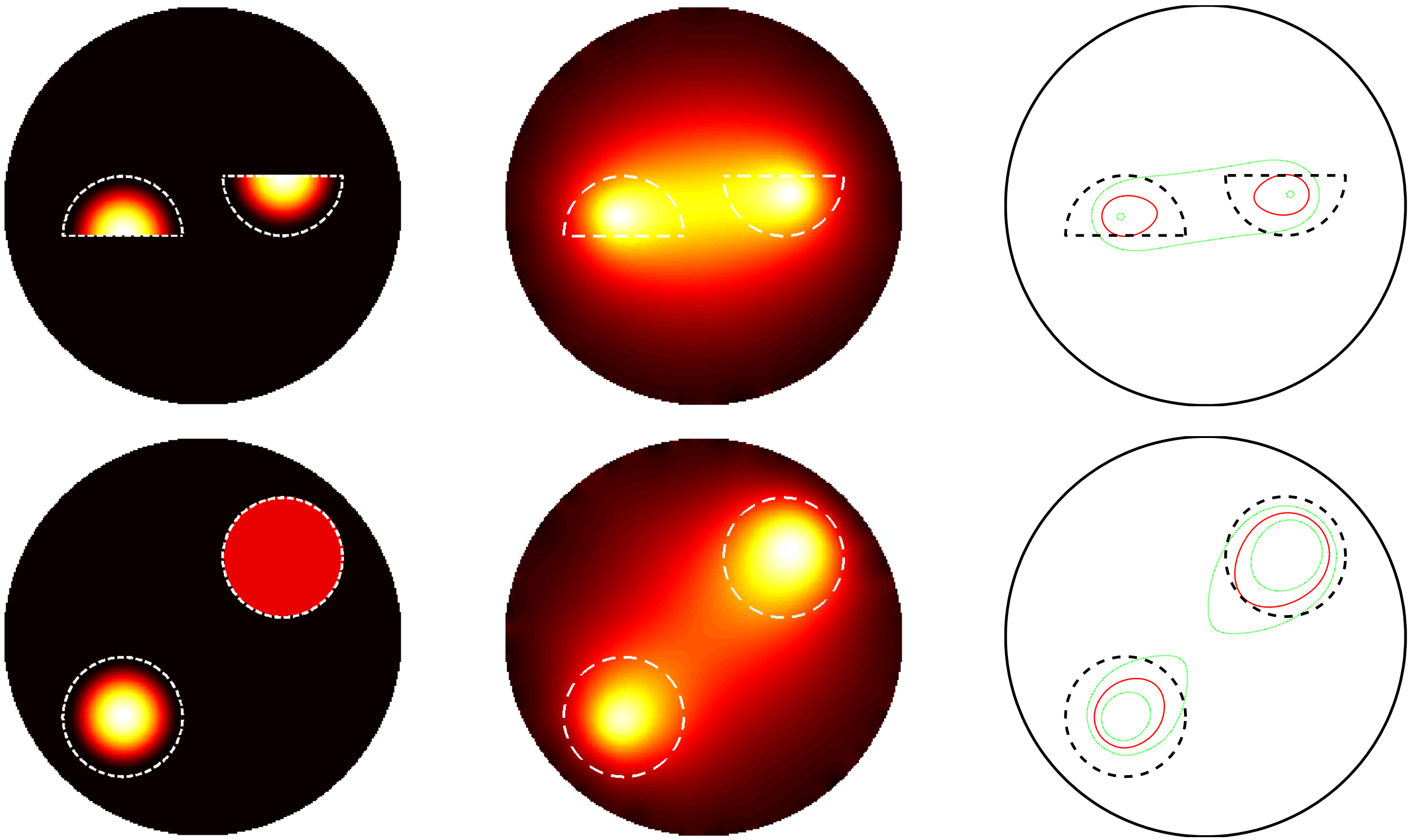
Numerical example



real conductivity $\quad$ "$\| |\Lambda_0 - \Lambda_1|^{-1/2} \Phi_z \|$" $\quad$ contour lines

Numerical results for inclusions with smooth transitions in EIT

Numerical example



real conductivity $\quad$ "$\| |\Lambda_0 - \Lambda_1|^{-1/2} \Phi_z \|$" $\quad$ contour lines

Numerical results with 0.1% noise

Remarks

Remarks

- Using monotonicity arguments the assumptions of the Factorization Method can be reduced to local properties.
- FM also works for inclusions that are not sharply separated from the background.
- With the same technique one can eliminate boundary regularity assumptions, or simultaneously treat inclusions of different types (e.g. absorbing and conducting, G and Hyvönen 2008).

However,

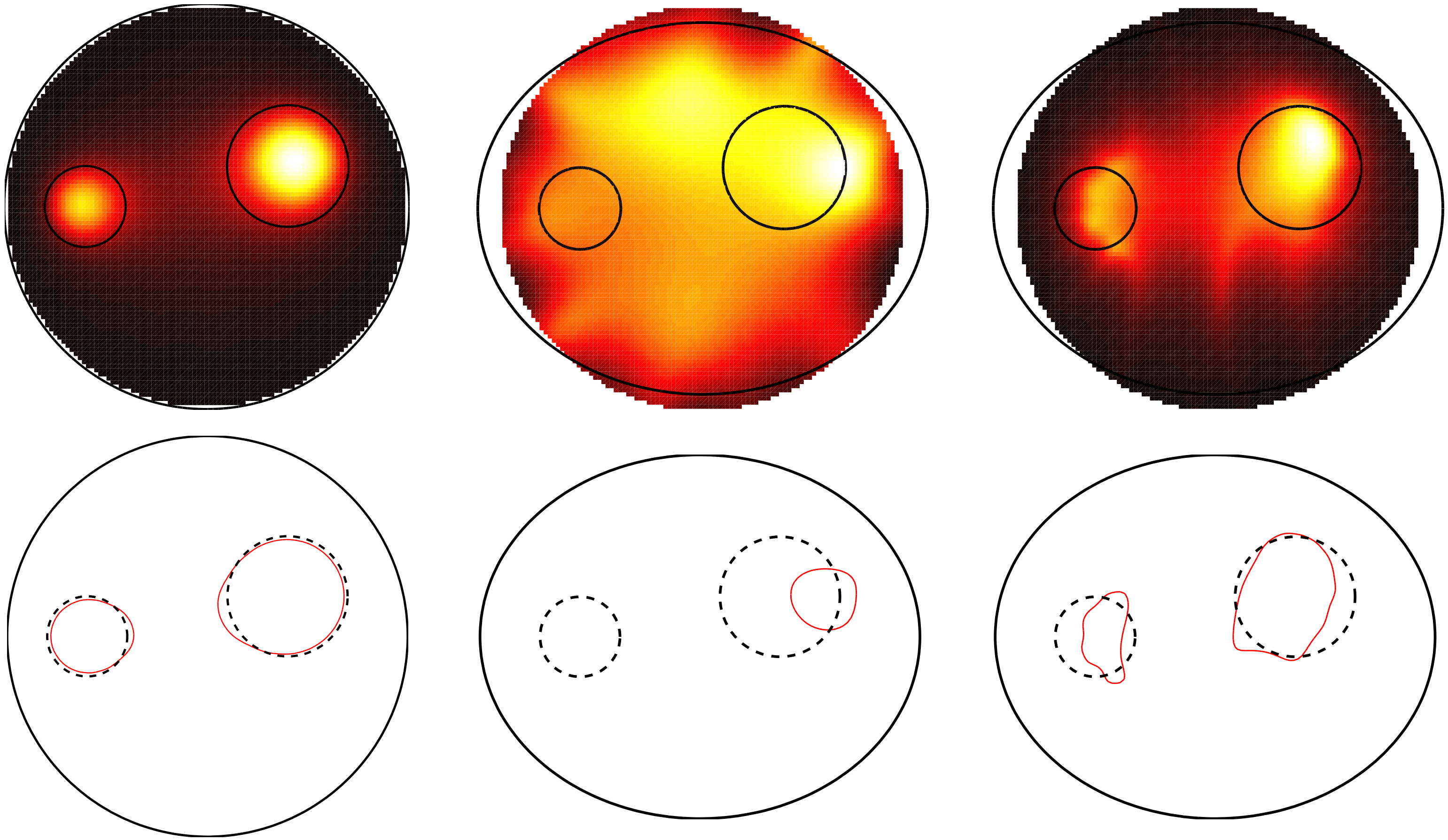
- The result still needs a global definiteness property ($\sigma_1 \geq 0$ or $\sigma_1 \leq 0$ in all inclusions). FM for indefinite problems is still an open problem.

Frequency-difference EIT

Reference measurements

- Factorization method uses difference $\Lambda_1 - \Lambda_0$ between
  - actual measurements Λ_1
  - reference measurements Λ_0 at an inclusion-free body
- Advantage: If reference measurements are available then systematic errors cancel out, e.g., forward modeling errors about the body geometry.
- Disadvantage: If reference measurements have to be simulated (or calculated analytically) then forward modeling errors have a large impact on the reconstructions.
- In medical application, reference measurements at an inclusion-free body are usually not available.

Example



Λ_1: circle
Λ_0: circle
FM for circle

Λ_1: ellipse
Λ_0: circle
FM for circle

Λ_1: ellipse
Λ_0: ellipse
FM for circle

Possible solution

- Possible solution: Replace reference measurements by measurements at another frequency.
- Frequency-dependent EIT:
  - g: applied current, time-harmonic with frequency ω

  $\rightsquigarrow$ electric potential u^ω that solves

$$\nabla \cdot \gamma^\omega \nabla u^\omega = 0, \qquad \gamma^\omega \partial_\nu u^\omega|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else.} \end{cases}$$

  with complex conductivity $\gamma^\omega = \sigma + \mathrm{i}\epsilon\omega$, ϵ: dielectricity.
- Measurements at frequency $\omega \rightsquigarrow$ Current-to-voltage (NtD) map

$$\Lambda^\omega : \ g \mapsto u^\omega|_S$$

Sketch of the idea

How to use frequency-difference measurements:

- Given two frequ. $\omega, \tau > 0$, conductivities $\gamma^\omega, \gamma^\tau$ and NtDs Λ^ω, Λ^τ
- Assume that for all x outside the inclusion Ω

$$\gamma^\omega(x) = \gamma_0^\omega \in \mathbb{C} \quad \text{and} \quad \gamma^\tau(x) = \gamma_0^\tau \in \mathbb{C}$$

- Using $\gamma_0^\omega \Lambda_\omega$ and $\gamma_0^\tau \Lambda_\tau$ scales down conductivity outside Ω to 1.

$\rightsquigarrow$ Difference $\gamma_0^\omega \Lambda_\omega - \gamma_0^\tau \Lambda_\tau$ should have similar properties to $\Lambda_1 - \Lambda_0$.

FM should also work with $\gamma_0^\omega \Lambda_\omega - \gamma_0^\tau \Lambda_\tau$ instead of $\Lambda_1 - \Lambda_0$.

- For non-zero frequencies, $\gamma_0^\omega \Lambda_\omega$ is not self-adjoint, so we will have to use its real or imaginary part

$$\Im(A) := \frac{1}{2\mathrm{i}}(A - A^*), \qquad \Re(A) := \frac{1}{2}(A + A^*)$$

for an operator $A : \; L^2_\diamond(S) \to L^2_\diamond(S)$.

fdEIT

Theorem *(G, Seo 2008)*

Let Ω have a connected complement,

$$\gamma^\omega(x) = \gamma_0^\omega + \gamma_\Omega^\omega(x)\chi_\Omega(x), \quad \text{and} \quad \gamma^\tau(x) = \gamma_0^\tau + \gamma_\Omega^\tau(x)\chi_\Omega(x).$$

- If $\Im\left(\frac{\gamma_\Omega^\omega}{\gamma_0^\omega}\right) \in L_+^\infty(\Omega)$ or $-\Im\left(\frac{\gamma_\Omega^\omega}{\gamma_0^\omega}\right) \in L_+^\infty(\Omega)$, then

$$z \in \Omega \quad \text{if and only if} \quad \Phi_z|_{\partial B} \in \mathcal{R}\left(\left|\Im\left(\sigma_0^\omega \Lambda_\omega\right)\right|^{1/2}\right),$$

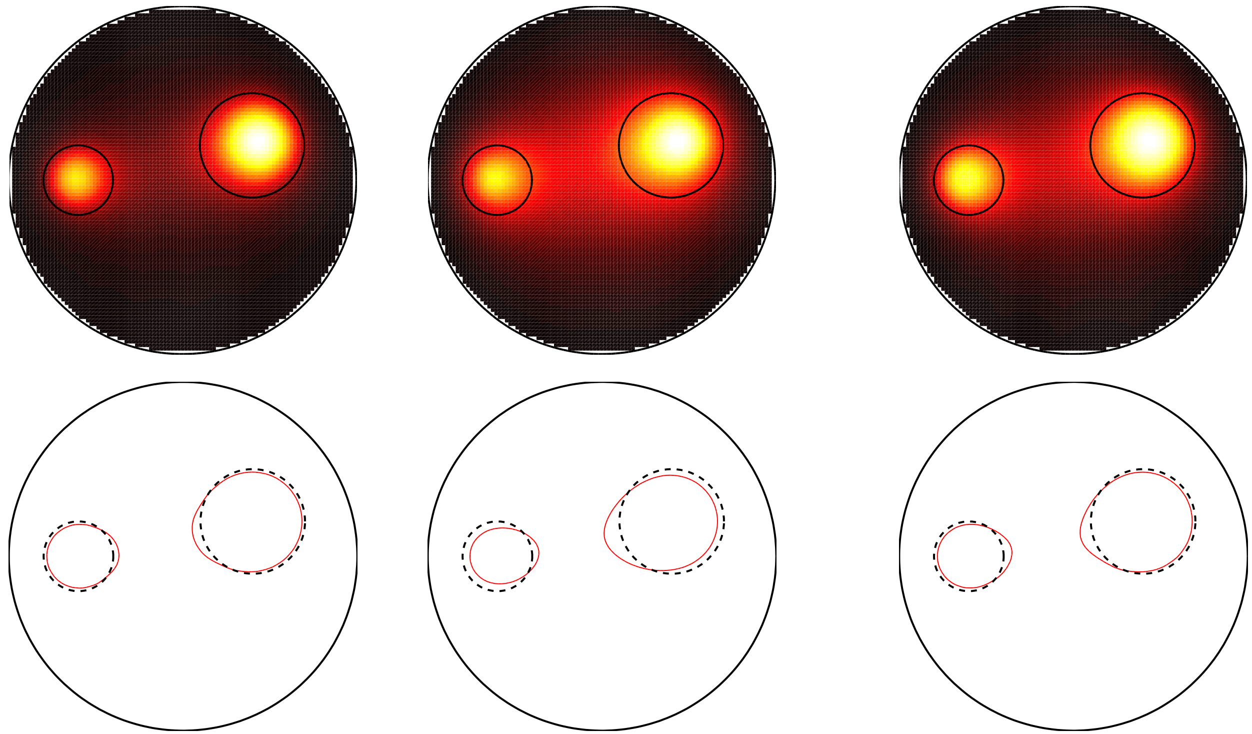
- If $\Re\left(\frac{\sigma_\Omega^\tau}{\sigma_0^\tau}\right) - \Re\left(\frac{\sigma_\Omega^\omega}{\sigma_0^\omega}\right) - \frac{\Im\left(\frac{\sigma_\Omega^\omega}{\sigma_0^\omega}\right)^2}{\Re\left(\frac{\sigma_\Omega^\omega}{\sigma_0^\omega}\right)} \in L_+^\infty(\Omega)$, then

$$z \in \Omega \quad \text{if and only if} \quad \Phi_z|_{\partial B} \in \mathcal{R}\left(\left|\Re\left(\sigma_0^\omega \Lambda_\omega - \sigma_0^\tau \Lambda_\tau\right)\right|^{1/2}\right).$$

($\tau = 0$ possible and same assertion also holds with interchanged ω and τ).

FM can be used on single non-zero frequency data or on frequency-difference data.

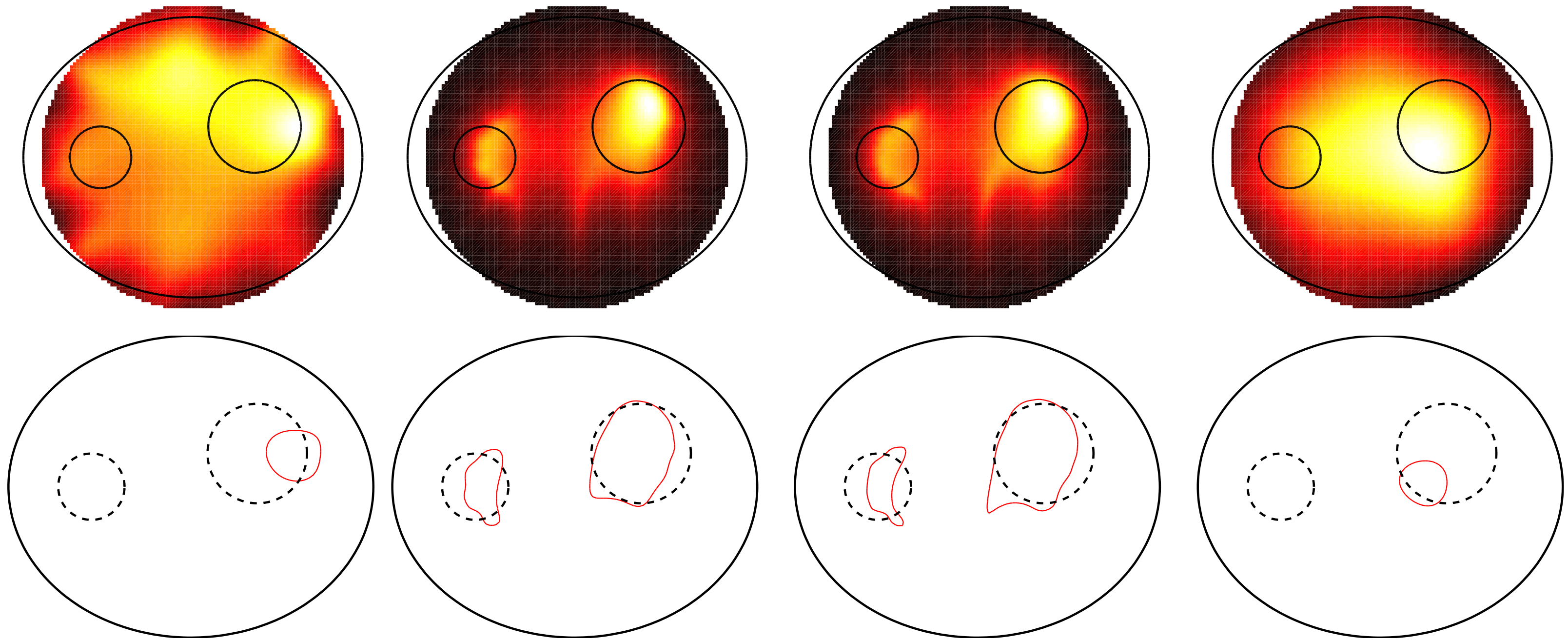
Numerical example



FM with $|\Lambda_0 - \Lambda_1|^{1/2}$ | FM with $|\Im(\sigma_0^\omega \Lambda_\omega)|^{1/2}$ | FM with $|\Re(\sigma_0^\omega \Lambda_\omega - \Lambda_1)|^{1/2}$

(Conductivities: $\sigma = 0.3 - 0.2\chi_\Omega(x), \quad \sigma_\omega(x) = 0.3 + 0.1\mathrm{i} - 0.2\chi_\Omega(x)$.)

Numerical example



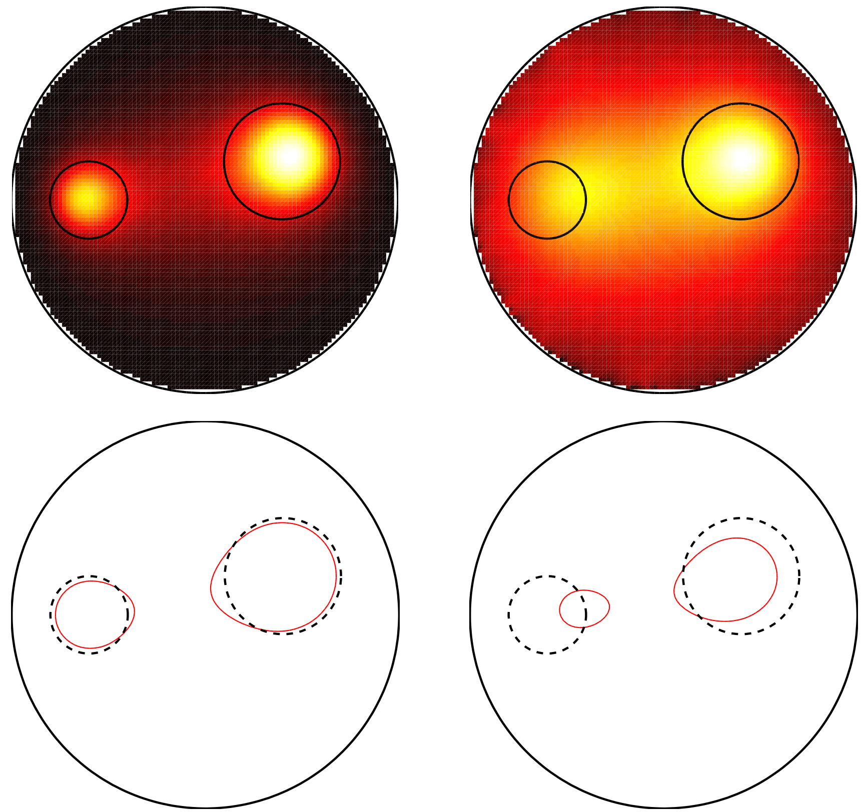
$|\Lambda_0^{\text{circ}} - \Lambda_1|^{\frac{1}{2}}$ $|\Lambda_0^{\text{ellipse}} - \Lambda_1|^{\frac{1}{2}}$ $|\Re\left(\sigma_0^{\omega}\Lambda_{\omega} - \Lambda_1\right)|^{\frac{1}{2}}$ $|\Im\left(\sigma_0^{\omega}\Lambda_{\omega}\right)|^{\frac{1}{2}}$

Reconstructions of an ellipse-shaped body that is wrongly assumed to be a circle.

Unknown background

- FM for frequency-difference EIT requires no reference data but still needs to know the constant background conductivity
- Heuristic method to estimate this from the data:
  - Eigenvectors for low eigenvalues should belong to highly oscillating potentials that do not penetrate deeply.
  - Most of the quotients of eigenvalues of Λ^ω and Λ^τ should behave like $\gamma_0^\tau/\gamma_0^\omega$.
  - For zero-frequency data $\Lambda^\tau = \Lambda_1$ we use $\left|\Re\left(\alpha\Lambda_\omega - \Lambda_1\right)\right|^{\frac{1}{2}}$ with the median α of quotients of eigenvalues of Λ^ω and Λ_1.
  - Analogously, the phase of γ_0^ω can be estimated from quotients of real and imaginary part of the eigenvalues of Λ^ω.

Unknown background



no noise $\qquad$ 0.1% noise

Reconstructions for unknown background conductivity without and with noise.

Remarks

Remarks

- Simulating reference data makes Factorization Method vulnerable to forward modeling errors.
- Using frequency-difference measurements strongly improves FMs robustness. Results are comparable to those with correct reference data.
- Unknown background conductivities can be estimated from the data.

However,

- Scaling the conductivity by simple multiplication only works for constant background conductivity.
- Theory needs contrast conditions in the inclusions with global definiteness properties.

Conclusions

Comparatively simple monotony arguments yield :

- New theoretical identifiability result for the Calderon-problem
  - *Two conductivities can be distinguished if one is larger in some part that can be connected to the boundary without crossing a sign change.*
- Improvements for the Factorization Method:
  - *FM does not only find inclusions where a parameter “jumps”, but also where it merely “differs” from a known background value.*
  - *Reference data can be replaced by frequency-difference data, thus strongly improving the methods robustness.*

However,

- There are important open problems connected with definiteness properties.