Sampling methods for low-frequency electromagnetic imaging

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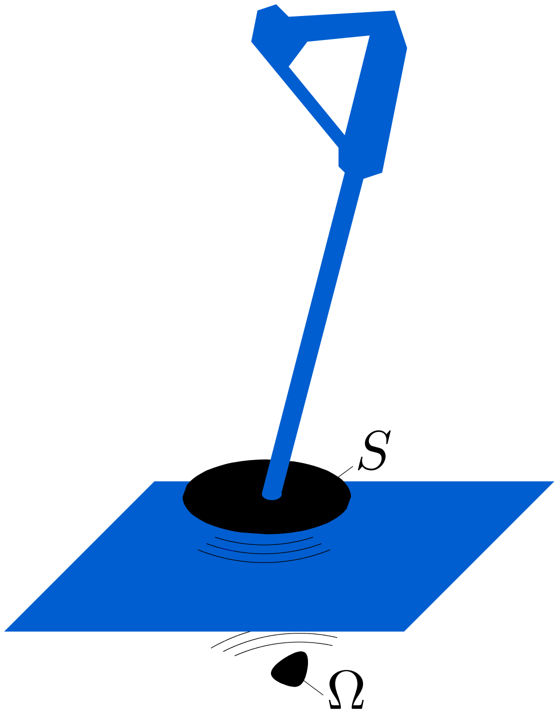
Oberwolfach workshop on Inverse Problems in Wave Scattering,

Oberwolfach, March 4–10, 2007

Setting



S: Measurement device

Ω: Magnetic / dielectric object

- Apply surface currents J on S (time-harmonic with frequency ω).

$\rightsquigarrow$ Electromagnetic field (E^ω, H^ω) (time-harmonic with frequency ω)

- Measure field on S (and try to locate Ω from it).

Idealistic assumption:

- Measure (tangential component of) $E^\omega|_S$ for all possible J

$\rightsquigarrow$ Measurement operator: $M^\omega : J \mapsto \gamma_\tau E^\omega|_S$

Goal: Locate Ω from the measurements M^ω.

Maxwell's equations

Time-harmonic Maxwell's equations

$$\begin{aligned} \operatorname{curl} H^\omega + \mathrm{i}\,\omega\epsilon E^\omega &= J && \text{in } \mathbb{R}^3, \\ -\operatorname{curl} E^\omega + \mathrm{i}\,\omega\mu H^\omega &= 0 && \text{in } \mathbb{R}^3. \end{aligned}$$

Silver-Müller radiation condition (RC)

$$\int_{\partial B_\rho} \left| \nu \wedge \sqrt{\mu} H^\omega + \sqrt{\epsilon} E^\omega \right|^2 \mathrm{d}\sigma = o(1), \quad \rho \to \infty.$$

E^ω:	electric field	ϵ:	dielectricity
H^ω:	magnetic field	μ:	permeability
ω:	frequency	J:	applied currents, $\operatorname{supp} J \subseteq S$

More idealistic assumptions: $\epsilon = 1$, $\mu = 1$ outside the object Ω

Typical metal detectors work at very low frequencies:

frequency ≈ 20kHz, wavelength ≈ 15km, $\omega \approx 4 \times 10^{-4}$m^{-1}

Forward Problem

Eliminate H^ω from Maxwell's equations:

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} E^\omega - \omega^2 \epsilon E^\omega = \mathrm{i}\omega J \quad \text{in } \mathbb{R}^3, \tag{1}$$

$$+ \text{ radiation condition.} \tag{RC}$$

Function space: $E^\omega \in H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3; \mathbb{C}^3)$

$\rightsquigarrow$ $\begin{cases} \text{Left side of (1) makes sense (in } \mathcal{D}'(\mathbb{R}^3;\mathbb{C}^3)), \\ E^\omega \text{ has tangential trace on } S\text{: } \gamma_t E^\omega|_S \in TH^{-1/2}(\operatorname{curl}, S). \end{cases}$

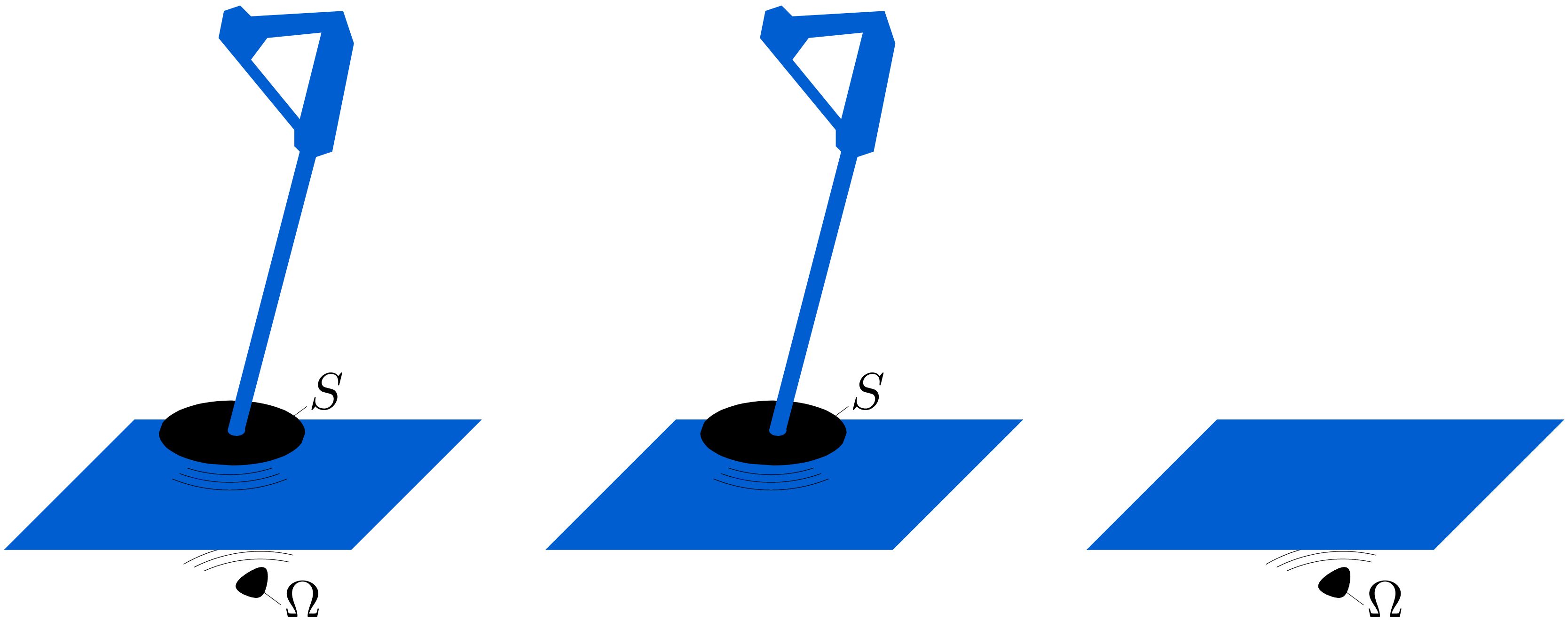
Under certain conditions (1)+(RC) have a unique solution for all

$$J \in TH^{-1/2}(\operatorname{div}, S) = TH^{-1/2}(\operatorname{curl}, S)'$$

and the solution depends continuously on J.

$\rightsquigarrow$
$$M^\omega : \ TH^{-1/2}(\operatorname{div}, S) \to TH^{-1/2}(\operatorname{curl}, S), \quad J \mapsto \gamma_\tau E^\omega|_S$$
is a continuous, linear operator.

Scattered Field



$$M_t^\omega : \; J \mapsto \gamma_\tau E_t^\omega,$$

E_t^ω solution for

$$\epsilon = 1 + \epsilon_1 \chi_\Omega(x)$$
$$\mu = 1 + \mu_1 \chi_\Omega(x)$$

"total field"

$$M_i^\omega : \; J \mapsto \gamma_\tau E_i^\omega,$$

E_i^ω solution for

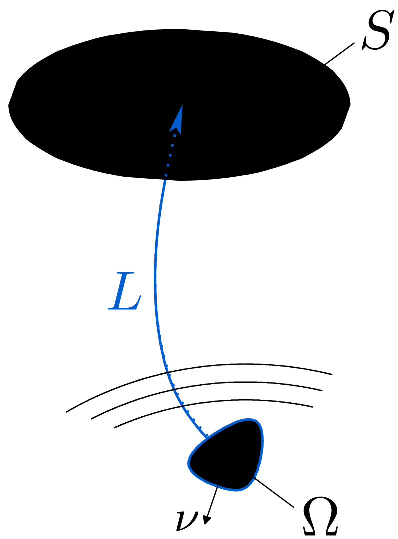
$$\epsilon = 1, \quad \mu = 1$$

"incoming field"

$$M_s^\omega := M_t^\omega - M_i^\omega$$

"scattered field"

Virtual Measurements



ψ: (tang. comp. of) a magnetic field on $\partial\Omega$

$$L: \begin{cases} TH^{-1/2}(\mathrm{div}, \partial\Omega) \to TH^{-1/2}(\mathrm{curl}, S), \\ \psi \mapsto \gamma_\tau E^\omega, \end{cases}$$

where

$$\mathrm{curl}\,\mathrm{curl}\, E^\omega - \omega^2 E^\omega = 0 \qquad \text{in } \mathbb{R}^3,$$
$$\nu \wedge \mathrm{curl}\, E^\omega|_{\partial\Omega} = \psi \qquad + \text{(RC)}.$$

L is part of the measurement operator:

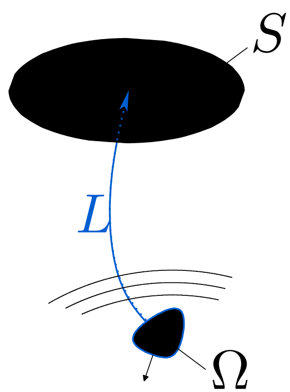
$$M_s^\omega = LG,$$

with $G: \; J \mapsto \nu \wedge \mathrm{curl}\, E_s^\omega|_{\partial\Omega}$.

Actually $M_s^\omega = LFL^T$

(cf. G., Hanke, Kirsch, Muniz, Schneider (2005)
for magnetic excitations and perfectly conducting objects).

Range characterization



Electric field of a point current in point z with direction d:

$$\operatorname{curl}\operatorname{curl} E^{\omega}_{z,d} - \omega^2 E^{\omega}_{z,d} = \mathrm{i}\omega\delta_z d \quad \text{in } \mathbb{R}^3$$

+ (RC).

$\mathcal{R}(L)$ determines Ω:

For every z below S and every direction d

$$z \in \Omega \quad \Longleftrightarrow \quad \gamma_\tau E^{\omega}_{z,d} \in \mathcal{R}(L)$$

$\Longrightarrow$: If $z \in \Omega$ then $E^{\omega}_{z,d}$ solves eqs. in definition of L.

$\Longleftarrow$: If $z \notin \overline{\Omega}$ then every function that "looks like" $E^{\omega}_{z,d}$ on S must have a singularity in z because of analytic continuation.

(G., Hanke, Kirsch, Muniz, Schneider, 2005)

Linear Sampling Method (LSM)

$$\gamma_\tau E^\omega_{z,d} \in \mathcal{R}(M^\omega_s) = \mathcal{R}(LG) \subseteq \mathcal{R}(L) \quad \Longrightarrow \quad z \in \Omega$$

Linear Sampling Method (Colton, Kirsch 1996):

- For every z below S test whether $\gamma_\tau E^\omega_{z,d} \in \mathcal{R}(M^\omega_s)$.

⇝ (LSM) finds a subset of Ω.

More common formulation:

- M^ω_s has dense range.

⇝ For all $\epsilon > 0$ there exists $J_{z,\epsilon}$ with $\|M^\omega_s J_{z,\epsilon} - \gamma_\tau E^\omega_{z,d}\| < \epsilon$.

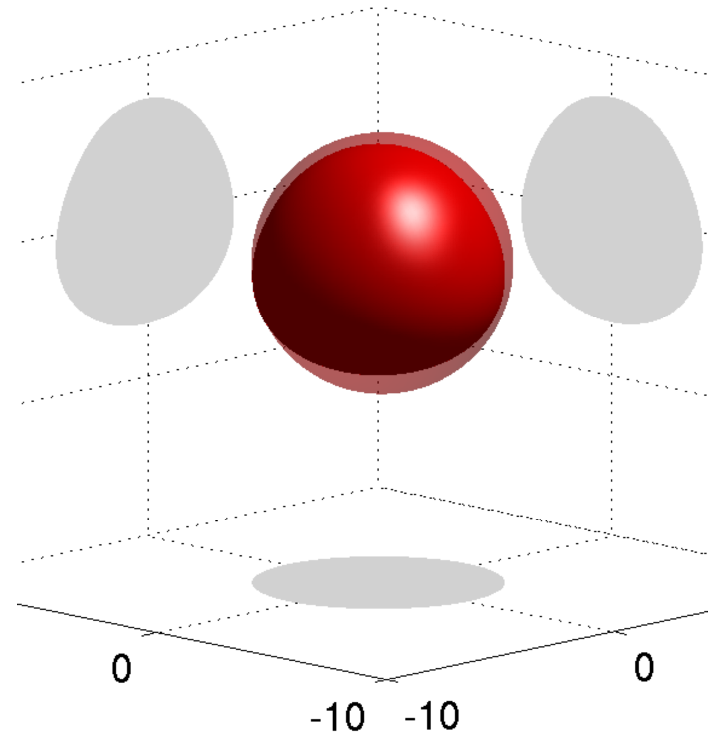
- M^ω_s compact

⇝ If $z \notin \overline{\Omega}$ then $\gamma_\tau E^\omega_{z,d} \notin \mathcal{R}(M^\omega_s)$ and thus $\|J_{z,\epsilon}\| \to \infty$ for $\epsilon \to 0$.

⇝ For every $\epsilon > 0$ $J_{z,\epsilon}$ can be chosen such that $\lim_{z \to \partial\Omega} \|J_{z,\epsilon}\| = \infty$.

Numerical result (by C. Schneider)

- Size of S: $32\text{cm} \times 32\text{cm}$
- $\omega = 4 \times 10^{-4}$, i. e. frequency $\approx 20\text{kHz}$
- Neumann-boundary condition
  $$\nu \wedge \operatorname{curl} E|_{\partial\Omega} = 0$$
  ("infinite permeability in Ω")
- Currents imposed / electric fields measured on a 6×6 grid on S
- Ω: ball with $r = 4\text{cm}$, 15cm below S
- Forward solver:
  BEM from Erhard / Potthast, Göttingen



Theory: LSM finds a (possibly empty) subset of Ω

Numerics: LSM finds Ω

Why are the results so good?

Factorization Method

$$\gamma_\tau E^\omega_{z,d} \in \mathcal{R}(L) \quad \Longleftrightarrow \quad z \in \Omega$$

LSM: $\mathcal{R}(M^\omega_s) \subseteq \mathcal{R}(L)$
$\rightsquigarrow$ Testing whether $\gamma_\tau E^\omega_{z,d} \in \mathcal{R}(M^\omega_s)$ finds subset of Ω.

For similar problems it was shown that

$$\mathcal{R}(L) = \mathcal{R}(|M^\omega_s|^{1/2}). \tag{2}$$

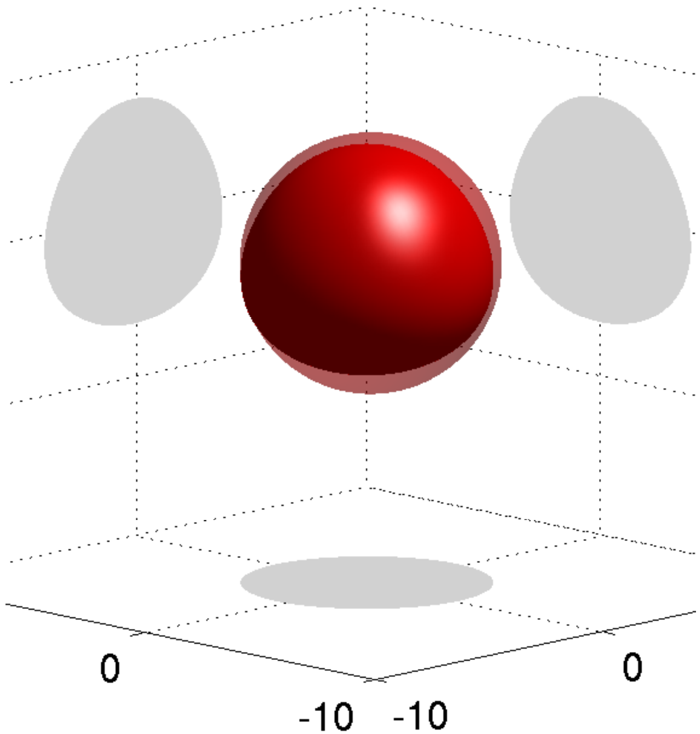
$\rightsquigarrow$ Testing whether $\gamma_\tau E^\omega_{z,d} \in \mathcal{R}(|M^\omega_s|^{1/2})$ finds Ω.
(so-called Factorization Method, Kirsch 1998).

Numerical examples show almost no difference between testing

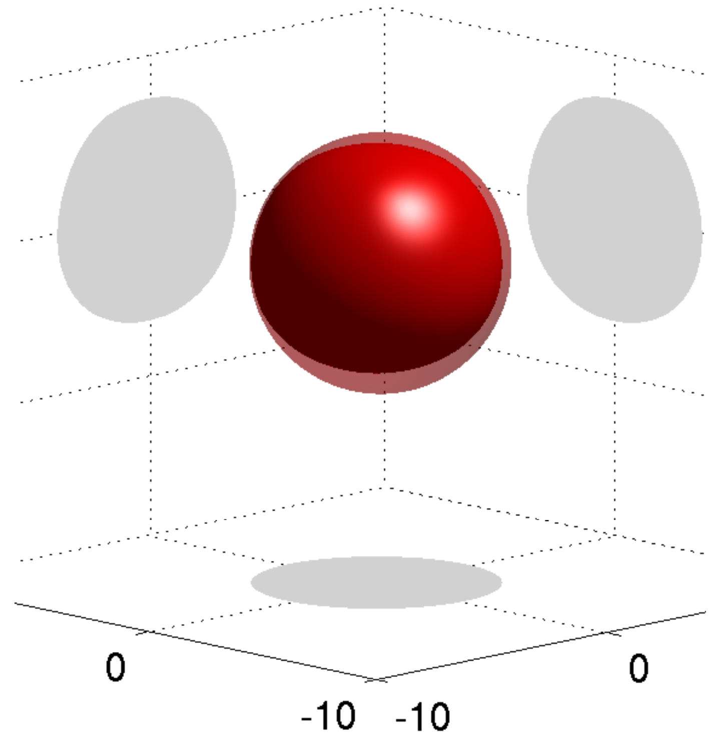
$$E^\omega_{z,d} \in \mathcal{R}(M^\omega_s) \quad \text{or} \quad E^\omega_{z,d} \in \mathcal{R}(|M^\omega_s|^{1/2}).$$

Possible explanation for the good performance: *maybe (2) holds*.

$\mathcal{R}(M_s^\omega)$ vs. $\mathcal{R}(|M_s^\omega|^{1/2})$



$E_{z,d}^\omega \in \mathcal{R}(M_s^\omega)$



$E_{z,d}^\omega \in \mathcal{R}(|M_s^\omega|^{1/2})$

Range identity

Range identity: $$\mathcal{R}(L) = \mathcal{R}(|M_s^\omega|^{1/2}). \qquad (2)$$

- originally developed by Kirsch (1998) for far field measurements for the Helmholtz equation,
- generalized to EIT (Brühl/Hanke, 1999), electrostatics (Hähner, 1999)
- holds for harmonic vector fields (Kress, 2002),
- holds for far-field measurements for Maxwell's equations (Kirsch, 2004)
- was shown for several other situations (Arens, Bal, Frühauf, Grinberg, Hyvönen, Schappel, Scherzer)
- holds for general real elliptic problems (G., 2006).

No result is known for this near-field measurements for Maxwell's eqs.

Low-frequency asymptotics

Maxwell's equations

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} E^\omega - \omega^2 \epsilon E^\omega = \mathrm{i}\omega J \quad \text{in } \mathbb{R}^3$$

+ radiation condition (RC)

also imply

$$\operatorname{div}(\epsilon E^\omega) = \frac{1}{\mathrm{i}\omega} \operatorname{div} J \quad \text{in } \mathbb{R}^3$$

(Time-harmonic formulation of conservation of surface charges ρ

$$\operatorname{div} J = -\partial_t \rho, \qquad \operatorname{div}(\epsilon E^\omega) = \rho.)$$

Formal asymptotic analysis for $\operatorname{div} J \neq 0$:

$$E^\omega = \frac{1}{\omega} E_{-1} + E_0 + O(\omega),$$

Rigorous analysis (for fixed incoming waves): Ammari, Nédélec, 2000

(Low frequency electromagnetic scattering, SIAM J. Math. Anal.)

Formal asymptotic analysis

Asymptotic analysis: $E^\omega = \frac{1}{\omega} E_{-1} + E_0 + O(\omega)$, where E_{-1}, E_0 solve

$$\left.\begin{aligned} \operatorname{curl} \frac{1}{\mu} \operatorname{curl} E_{-1} &= 0, \\ \operatorname{div}(\epsilon E_{-1}) &= -\mathrm{i} \operatorname{div} J, \end{aligned}\right\} \rightsquigarrow \operatorname{curl} E_{-1} = 0 \rightsquigarrow E_{-1} = \nabla \varphi$$

$$\left.\begin{aligned} \operatorname{curl} \frac{1}{\mu} \operatorname{curl} E_0 &= 0, \\ \operatorname{div}(\epsilon E_0) &= 0, \end{aligned}\right\} \rightsquigarrow E_0 = 0$$

(ignoring radiation conditions)

$$\rightsquigarrow \quad \boxed{E^\omega = \frac{1}{\mathrm{i}\omega} \nabla \varphi + O(\omega), \quad \text{where } \operatorname{div}(\epsilon \nabla \varphi) = \operatorname{div} J.}$$

Interpretation:

$\frac{1}{\mathrm{i}\omega} \varphi$: electrostatic potential created by surface charges $\rho = \frac{1}{\mathrm{i}\omega} \operatorname{div} J$.

Electrostatic measurements

Consequence: The measurements

$$M_s^\omega : \ J \mapsto \gamma_\tau E_s^\omega$$

are essentially electrostatic measurements:

$$M_s^\omega \approx -\frac{1}{\mathrm{i}\omega} \nabla_S \Lambda_s \nabla_S^*, \qquad J \xmapsto{\nabla_S^*} \operatorname{div} J = \rho \xmapsto{\Lambda_s} \varphi|_S \xmapsto{\nabla_S} \gamma_\tau \nabla \varphi,$$

with the electrostatic measurement operator $\Lambda_s = \Lambda_t - \Lambda_i$,

$$\Lambda_t : \begin{cases} H^{-1/2}(S) \to H^{1/2}(S), \\ \rho \mapsto \varphi_t|_S, \end{cases} \qquad \Lambda_i : \begin{cases} H^{-1/2}(S) \to H^{1/2}(S), \\ \rho \mapsto \varphi_i|_S, \end{cases}$$

$$\operatorname{div}(\epsilon_t \nabla \varphi_t) = \rho \qquad \operatorname{div}(\epsilon_i \nabla \varphi_i) = 0$$

$$\epsilon_t = 1 + \epsilon_1 \chi_\Omega \qquad \epsilon_i = 1$$

"electrostatic measurements with object" — *"electrostatic measurements "without object"*

Factorization method for ES

For electrostatic measurements the range identity holds:

$$\mathcal{R}(|\Lambda_s|^{1/2}) = \mathcal{R}(L_{\mathrm{ES}})$$

with "virtual electrostatic measurements" $L_{\mathrm{ES}}: \; H^{-1/2}(\partial\Omega) \to H^{1/2}(S)$.

(Hähner 1999: grounded objects, G. 2006: general theory)

$$\rightsquigarrow \mathcal{R}(|\nabla_S \Lambda_s \nabla_S^*|^{1/2}) = \mathcal{R}(\nabla_S L_{\mathrm{ES}})$$

$E_{z,d}$: electrostatic field of a dipole in z with direction d
(= low-frequency limit of $E_{z,d}^{\omega}$)

For every z below S and every direction d

$$z \in \Omega \quad \Longleftrightarrow \quad \gamma_\tau E_{z,d} \in \mathcal{R}(\nabla_S L_{\mathrm{ES}}) = \mathcal{R}(|\nabla_S \Lambda_s \nabla_S^*|^{1/2})$$

$\rightsquigarrow$ *Factorization Method works for the low-frequency asymptotics.*

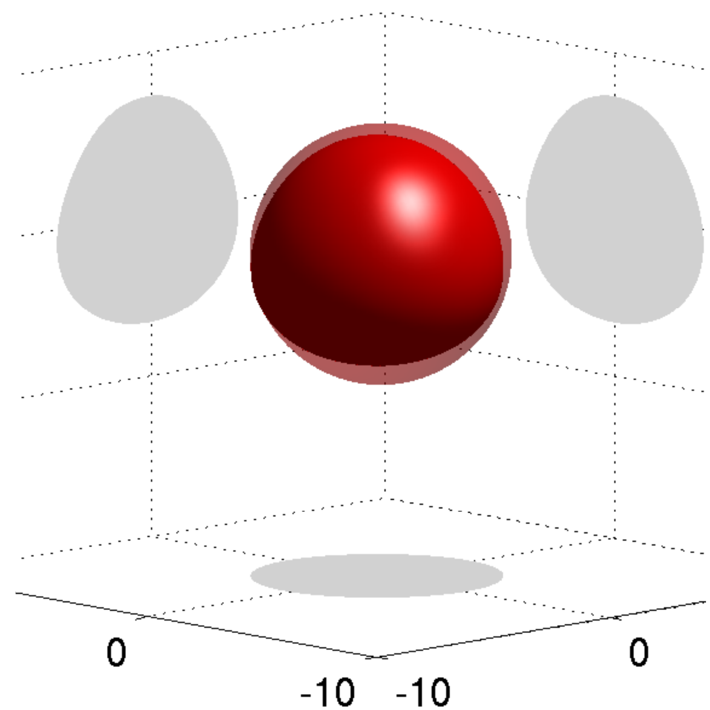
Consequences / Numerical result

Good performance of LSM can be explained by the fact that the measurements are (up to an error that is below measurement accuracy) electrostatic measurements for which the Factorization Method works.

$\rightsquigarrow$ *Treat data as electrostatic data and use Factorization Method.*

Numerical example

- Same forward data as in the previous example.
- Reconstruction used $E_{z,d}$ instead of $E^{\omega}_{z,d}$
- Boundary condition:
  $$\nu \wedge \operatorname{curl} E^{\omega}|_{\partial\Omega} = 0 \Longrightarrow \nu \cdot E^{\omega}|_{\partial\Omega} = 0$$
  ("zero dielectricity in Ω")



Current loops

- In practice: currents will be applied along closed loops.

$\rightsquigarrow$ $\operatorname{div} J = 0$, no electrostatic effects

- Also the electric field can only be measured along closed loops.

$\rightsquigarrow$ More realistic model for the measurements:

$$j^* M^\omega j,$$

where $j: TL^2_\diamond(S) = \{v \in TL^2(S), \operatorname{div} v = 0\} \hookrightarrow TH^{-1/2}(\operatorname{div}, S)$.

- j^* "factors out gradient fields", in particular

$$j^*(\gamma_\tau E_{z,d}) = 0 \quad \text{and} \quad j^*(\gamma_\tau E^\omega_{z,d}) \approx 0.$$

The presented sampling method relies on electrostatic effects that do not appear in practice.

Asymptotics again

Maxwell's equations

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} E^\omega - \omega^2 \epsilon E^\omega = \mathrm{i}\omega J, \qquad \operatorname{div}(\epsilon E^\omega) = 0.$$

+ *radiation condition (RC)*

Asymptotic analysis: $E_{-1} = E_0 = 0, \quad E^\omega = \omega E_1 + \omega^2 E_2 + \ldots$, with

$$\begin{aligned} \operatorname{curl} \frac{1}{\mu} \operatorname{curl} E_1 &= \mathrm{i}J, \\ \operatorname{div}(\epsilon E_1) &= 0, \\ \operatorname{curl} \frac{1}{\mu} \operatorname{curl} E_2 - \epsilon E_0 &= 0, \\ \operatorname{div}(\epsilon E_2) &= 0, \end{aligned} \quad \Bigg\} \rightsquigarrow E_2 = 0$$

(still ignoring additional conditions at $x = \infty$)

$\underset{\rightsquigarrow}{E := -\mathrm{i}E_1}$

$$E^\omega = \mathrm{i}\omega E + O(\omega^3), \text{ where} \quad \operatorname{curl} \frac{1}{\mu} \operatorname{curl} E = J, \quad \operatorname{div}(\epsilon E) = 0.$$

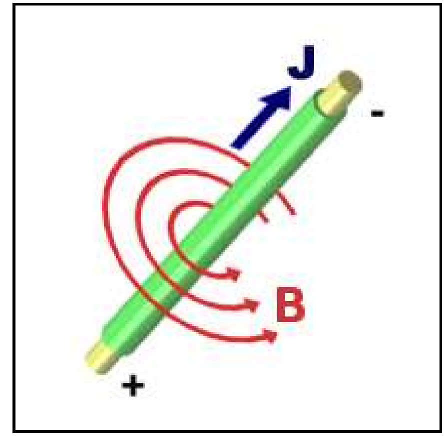
Interpretation

$$E^\omega = \mathrm{i}\omega E + O(\omega^3), \text{ where } \quad \operatorname{curl}\frac{1}{\mu}\operatorname{curl} E = J, \quad \operatorname{div}(\epsilon E) = 0.$$

- $B := \operatorname{curl} E$ solves

$$\operatorname{curl}\tfrac{1}{\mu}B = J, \quad \operatorname{div} B = 0.$$

$\rightsquigarrow$ B is the magnetostatic field generated by a steady current J (*Ampère's Law*).



- $B = \frac{1}{\mathrm{i}}\operatorname{curl} E \quad \rightsquigarrow \quad E$ is a vector potential of B (unique up to addition of A with $\operatorname{curl} A = 0$, i. e. up to $A = \nabla\varphi$).
- $\operatorname{div}(\epsilon E) = 0$ determines E uniquely (so-called *Coulomb gage).*

$\rightsquigarrow$ *E is (a potential of) the magnetostatic field induced by J.*

Figure based on `http://de.wikipedia.org/wiki/Bild:RechteHand.png`*, published under the GNU Free Documentation License (FDL) by "Frau Holle".*

Rigorous analysis

Theorem

For suff. small ω there exists a unique E^ω of Maxwell's equations and for every ball B_r there exists $C_r > 0$ such that

$$\|E^\omega - \mathrm{i}\omega E\|_{H(\mathrm{curl},B_r)} \leq C_r \omega^3 \|J\|_{TL^2_\diamond(S)}.$$

Proof

- Reduce Maxwell's equation and magnetostatic equations both to a bounded domain B_r with non-local exact boundary conditions.
- For low frequencies the reduced problems are equivalent to new variational formulations containing also the normal component of the field.
- These new variational formulations satisfy a uniform coerciveness condition from which the unique solvability of Maxwell's equations and the asserted asymptotics follow.

Magnetostatic measurements

Analogously to the electrostatic case, the measurements

$$j^* M_s^\omega j : \; J \mapsto \gamma_\tau E_s^\omega$$

are now essentially magnetostatic measurements:

$$j^* M_s^\omega j \approx -\mathrm{i}\omega M_s,$$

with the magnetostatic measurement operator $M_s = M_t - M_i$,

$$M_t : \begin{cases} TL^2_\diamond(S) \to TL^2_\diamond(S)', \\ J \mapsto \gamma_\tau E_t|_S, \end{cases} \qquad M_i : \begin{cases} TL^2_\diamond(S) \to TL^2_\diamond(S)', \\ J \mapsto \gamma_\tau E_i|_S, \end{cases}$$

$$\begin{aligned} \operatorname{curl} \frac{1}{\mu_t} \operatorname{curl} E_t &= J \\ \operatorname{div} \; E_t &= 0 \\ \mu_t &= 1 + \mu_1 \chi_\Omega \end{aligned} \qquad \begin{aligned} \operatorname{curl} \frac{1}{\mu_i} \operatorname{curl} E_i &= J \\ \operatorname{div} \; E_i &= 0 \\ \mu_i &= 1 \end{aligned}$$

"magnetostatic measurements with object" *"magnetostatic measurements "without object"*

(Note that replacing $\operatorname{div} \epsilon E = 0$ *with* $\operatorname{div} E = 0$ *changes* E *only by a gradient field.)*

Factorization Method for MS

For magnetostatic measurements the range identity holds:

$$\mathcal{R}(|M_s|^{1/2}) = \mathcal{R}(L_{\mathrm{MS}})$$

with "virtual magnetostatic measurements"

$$L_{\mathrm{MS}} : \; TH^{-1/2}(\partial\Omega)/N \to TL^2_\diamond(S)'.$$

(Kress, 2002: similar situation with harmonic vector fields, G. 2006: general theory)

$G_{z,d} = \operatorname{curl} \frac{d}{|x-z|}$: vector potential of the magnetostatic field of a magnetic dipole in z with direction d

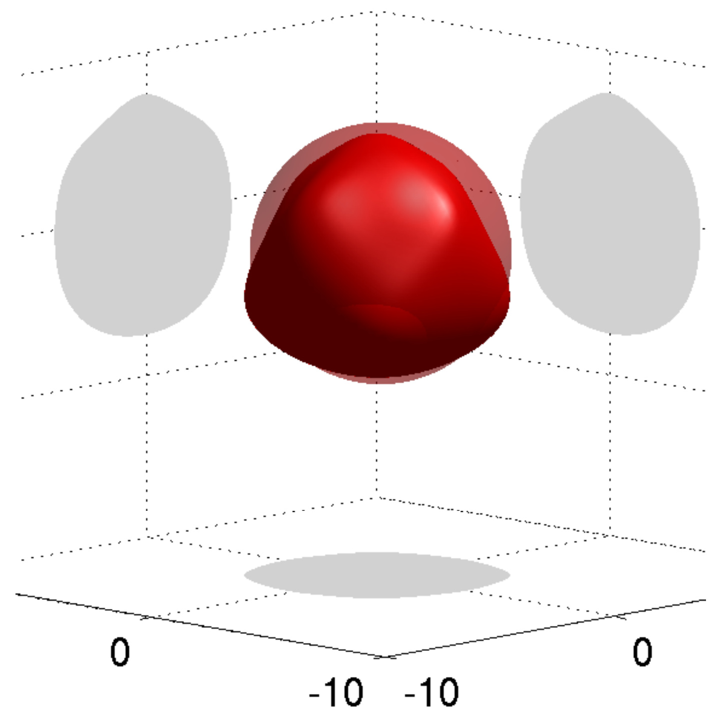
For every z below S and every direction d

$$z \in \Omega \quad \Longleftrightarrow \quad \gamma_\tau G_{z,d} \in \mathcal{R}(L_{\mathrm{MS}}) = \mathcal{R}(|M_s|^{1/2})$$

Factorization Method also works for (the low-frequency asymptotics of) current loops but a different singular function has to be used.

Numerical result

- Same setting as in the previous two examples.
- Simulated divergence-free currents (current loops) using normal magnetic dipole excitation on a 12×12 grid.
- Dirichlet boundary condition
$$\nu \wedge E|_{\partial\Omega} = 0$$
("perfectly conducting object")
- Reconstruction using $\gamma_\tau G_{z,d}$, i. e. with the Factorization Method for magnetostatic data.



Eddy currents

What happens if the object has a finite conductivity $\sigma > 0$?

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} E^\omega - \omega^2 \epsilon E^\omega = \mathrm{i}\omega(J + \sigma E^\omega)$$

Low frequency asymptotics in the time domain lead to

$$\partial_t(\sigma E) - \operatorname{curl} \frac{1}{\mu} \operatorname{curl} E = -\partial_t J,$$

which is parabolic in the object ($\sigma > 0$) and elliptic outside ($\sigma = 0$).
(Ammari, Buffa, Nédélec, 2000, SIAM J. Math. Anal.)

For the scalar model problem

$$\partial_t(\chi_\Omega u) - \operatorname{grad} \kappa \operatorname{div} u = 0$$

the Factorization Method works for $\kappa = 1 + \kappa_1 \chi_\Omega$ with $\kappa_1 > 0$.
(*Frühauf, G., Scherzer, to appear in SIAM J. Numer. Anal.)*

We expect that the method also works for conducting diamagnetic objects (e. g. copper).

Summary / Conclusions

For low-frequency electromagnetic imaging we obtained

- "Afterward explanation:"

  Good performance of the LSM for near-field Maxwell's equations can be explained by the fact that the measurements (essentially) agree with electrostatic measurements for which the Factorization Method works.

- "Practical consequence:"

  For the practically relevant case of current loops the test function for sampling methods should be replaced by $G_{z,d} = \operatorname{curl} \frac{d}{|x-z|}$.
  In other words, one should consider the measurements as magnetostatic measurements and apply the Factorization Method.

- "Optimistic outlook:"

  Analysis of a scalar model problem suggests that the method also works for conducting diamagnetic objects (e. g. copper).