Detecting Interfaces in a Parabolic-Elliptic Problem

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A parabolic-elliptic problem

Heat equation:

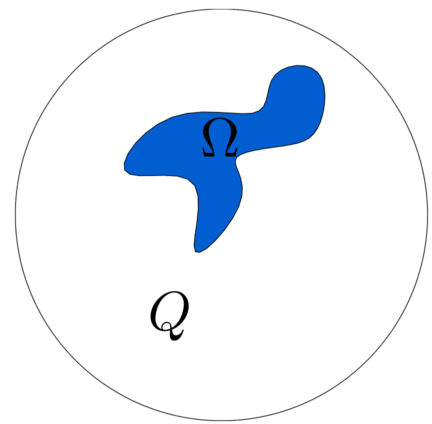
$$\partial_t(c(x)u(x,t)) - \nabla \cdot (\kappa(x)\nabla u(x,t)) = 0, \quad (x,t) \in B \times]0,T[$$

$u(x,t)$: temperature $\qquad c(x)$: heat capacity $\qquad \kappa(x)$: heat conductivity

Special case: $c = 0$ on a subset of B

e. g. $B = Q \cup \overline{\Omega}$,

$$c(x) = \chi_\Omega(x) = \begin{cases} 1 & \text{in } \Omega \\ 0 & \text{in } Q \end{cases}$$



$$\rightsquigarrow \begin{cases} \text{Parabolic equation in } \Omega & (\kappa = 1: \quad \partial_t u - \Delta u = 0) \\ \text{Elliptic equation in } Q & (\kappa = 1: \quad \Delta u = 0) \end{cases}$$

Motivation / Applications

- Inclusions of high heat capacity in domain with negligible capacity
- Ammari, Buffa, Nédélec (2000):

  Scattering of low-frequency electromagnetic waves by an electrically conducting object Ω in a non-conducting domain:

$$\partial_t(\sigma E) - \operatorname{rot} \frac{1}{\mu} \operatorname{rot} E = -\partial_t J$$

σ:	conductivity,	μ:	permeability,
E:	electric field,	J:	applied currents

- Eddy currents in cylinder of infinite length:

$$\partial_t(\chi_\Omega u) - \Delta u = 0$$

($B = \mathbb{R}^2$, $\kappa = 1$: MacCamy, Suri (1987), Costabel (1990)
Costabel, Ervin, Stephan (1990))

Direct problem

Let $c(x) \in L^\infty(B)$, $c \geq 0$, $\kappa(x) \in L^\infty_+(B)$.

The left hand side of

$$\partial_t(cu) - \nabla \cdot (\kappa \nabla u) = 0$$

makes sense for every $u(x,t) \in H^{1,0}(B \times]0,T[)$.

A solution $u \in H^{1,0}(B \times]0,T[)$ of this equation has

- well-defined Neumann boundary values

$$\kappa \partial_\nu u|_{\partial B} \in H^{-\frac{1}{2},-\frac{1}{2}}(\partial B \times]0,T[).$$

  (If $c = 0$ around ∂B then $\kappa \partial_\nu u|_{\partial B} \in H^{-\frac{1}{2},0}(\partial B \times]0,T[)$.)

- well-defined initial values $\sqrt{c(x)}u(x,0) \in L^2(B)$.

Existence / Uniqueness

Theorem

For every

- heat capacity $c(x) \in L^\infty(B)$, $c \geq 0$, $c \not\equiv 0$
- heat conductivity $\kappa(x) \in L^\infty_+(B)$ and
- applied heat flux / Neumann data $g(x,t) \in H^{-\frac{1}{2},0}(\partial B \times]0,T[)$

there is a unique solution $u(x,t) \in H^{1,0}(B \times]0,T[)$ of

$$\begin{aligned} \partial_t(cu) - \nabla \cdot (\kappa \nabla u) &= 0, \\ \kappa \partial_\nu u|_{\partial B} &= g, \\ c(x)u(x,0) &= 0. \end{aligned}$$

Proof.
Space-time variational formulation and Lions Projection Lemma.

Sensitivity Analysis

Theorem

- Without additional smoothness assumptions:

  Let $c_n \to c$ in $L^\infty(B)$ and u_n, u be the corresponding solutions. Then $(u_n)_n$ has a subsequence that convergenges weakly against u.

- For a heat flux $g \in H^1(0,T,H^{1/2}(\partial B))$, $g(x,0) = 0$, the solution u depends continuously on the capacity $c \in L^\infty(B)$.

  Its (one-sided) directional derivative in a direction $d \in L^\infty(B)$

  ($\|d\| = 1$ with $c + hd \geq 0$ for suff. small h)

  is given by the solution v of

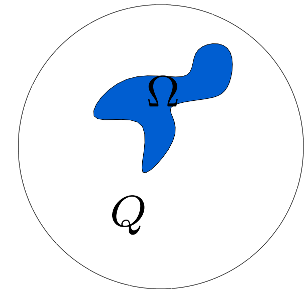
$$\partial_t(cv) - \nabla \cdot (\kappa \nabla v) = -d\dot{u}$$

  with zero initial and Neumann boundary data.

Proof.
Analysis of the space-time variational formulation.

Inverse Problem

Now: $c(x) = \chi_\Omega(x)$, $\kappa(x) = 1 + \kappa_1 \chi_\Omega$, $\kappa_1 > 0$,
$B = \overline{\Omega} \cup Q$, with unknown inclusion Ω.



Inverse Problem: Given a complete set of measurements

$$\Lambda_1 : \; g \mapsto u_1|_{\partial B}, \quad u_1 \text{ solves } \begin{cases} \partial_t(\chi_\Omega u_1) - \nabla \cdot (\kappa \nabla u_1) &= 0, \\ \partial_\nu u_1|_{\partial B} &= g, \\ u_1(x,0)|_\Omega &= 0. \end{cases}$$

reconstruct the interface $\partial\Omega$ resp. the inclusion Ω.

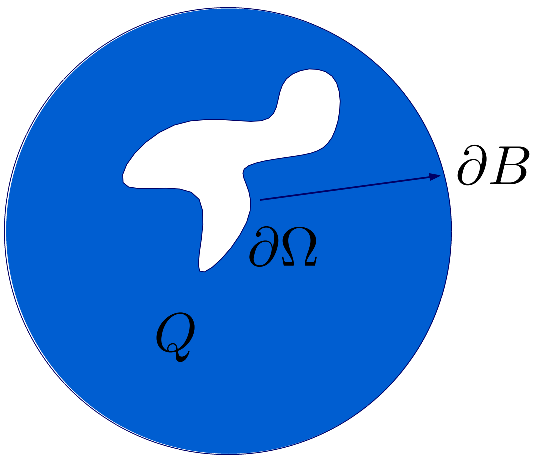
To solve the inverse problem we compare Λ_1 to reference measurements

$$\Lambda_0 : \; g \mapsto u_0|_{\partial B}, \quad u_0 \text{ solves } \Delta u_0 = 0, \quad \partial_\nu u_0|_{\partial B} = g,$$

i. e. measurements without an inclusion Ω.

Goal: Reconstruct Ω from given Λ_0 and Λ_1.

Virtual Measurements



ψ: given boundary flux on $\partial\Omega$

$$L: \ \psi \mapsto v|_{\partial B}, \qquad \text{where}$$

$$\Delta v(x,t) = 0 \quad \text{in } Q\times]0,T[, \tag{4}$$

$$\partial_\nu v|_{\partial B} = 0 \quad \text{on } \partial B, \tag{5}$$

$$\partial_\nu v|_{\partial\Omega} = \psi \quad \text{on } \partial\Omega. \tag{6}$$

$\mathcal{R}(L)$ determines Ω:

$$v_z|_{\partial B} \in \mathcal{R}(L) \quad \text{if and only if} \quad z \in \Omega$$

where v_z solves (4) in $B \setminus \{z\}$, v_z solves (5), v_z suff. singular in $z \in B$,

(e. g. a partial derivative of the Green's function for the Laplacian)

Factorization Method

Key identity of the so-called Factorization Method (for other problems!):

$$\mathcal{R}(L) = \mathcal{R}((\Lambda_0 - \Lambda_1)^{1/2}).$$

$\rightsquigarrow \mathcal{R}(L)$ (and thus Ω) can be computed from the measurements.

Such a range identity

- was originally developed by Kirsch for Inverse Scattering
- is known (under suitable conditions on the inclusion) for
  - Electrostatics (Hähner)
  - EIT (Brühl, Hanke), also with different electrode models (Brühl, Hanke, Hyvönen) and in the half space (Schappel)
  - Diffusion tomography (Kirsch), also with Robin B.C. (Hyvönen)
  - general real elliptic problems (G.)

Does a similar identity hold in this parabolic-elliptic case?

Range inclusions

- Range inclusions:

$$\mathcal{R}(\tilde{\Lambda}^{1/2}) \subseteq \mathcal{R}(L),$$

$$\mathcal{R}(\tilde{\Lambda}^{1/2}) \supseteq \mathcal{R}(L|_V),$$

$\tilde{\Lambda}$: symmetric part of $\Lambda_1 - \Lambda_0$,

V: space of boundary fluxes with certain temporal smoothness

- Existence of singular functions v_z with

$$v_z|_{\partial B} \in \mathcal{R}(L) \quad \text{if and only if} \quad z \in \Omega,$$

and $\partial_\nu v_z|_{\partial\Omega} \in V$.

$\rightsquigarrow$ $z \in \Omega$ if and only if $v_z|_{\partial B} \in \mathcal{R}(\tilde{\Lambda}^{1/2})$.

Sketch of the proof

$$\mathcal{R}(\tilde{\Lambda}^{1/2}) \subseteq \mathcal{R}(L),$$

$$\mathcal{R}(\tilde{\Lambda}^{1/2}) \supseteq \mathcal{R}(L|_V),$$

- Factorization:

$$\tilde{\Lambda} = LFL^*$$

- If $\|Ax\| \leq \|Bx\|$ for all x then $\mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$.

  $\rightsquigarrow \quad \mathcal{R}(\tilde{\Lambda}^{1/2}) = \mathcal{R}(LF^{1/2}) \subseteq \mathcal{R}(L)$.

- Coercivity condition for F

  $\rightsquigarrow \quad \mathcal{R}(F^{1/2}) \supseteq H^{\frac{1}{4}}(0,T,H_\diamond^{-\frac{1}{2}}(\partial\Omega)) =: V$.

Consequences / Remarks

- Theoretical result:

  $\partial\Omega$ is uniquely determined by Λ_1,

  i. e. the interface is uniquely determined by measuring all pairs of heat flux and temperature on ∂B.

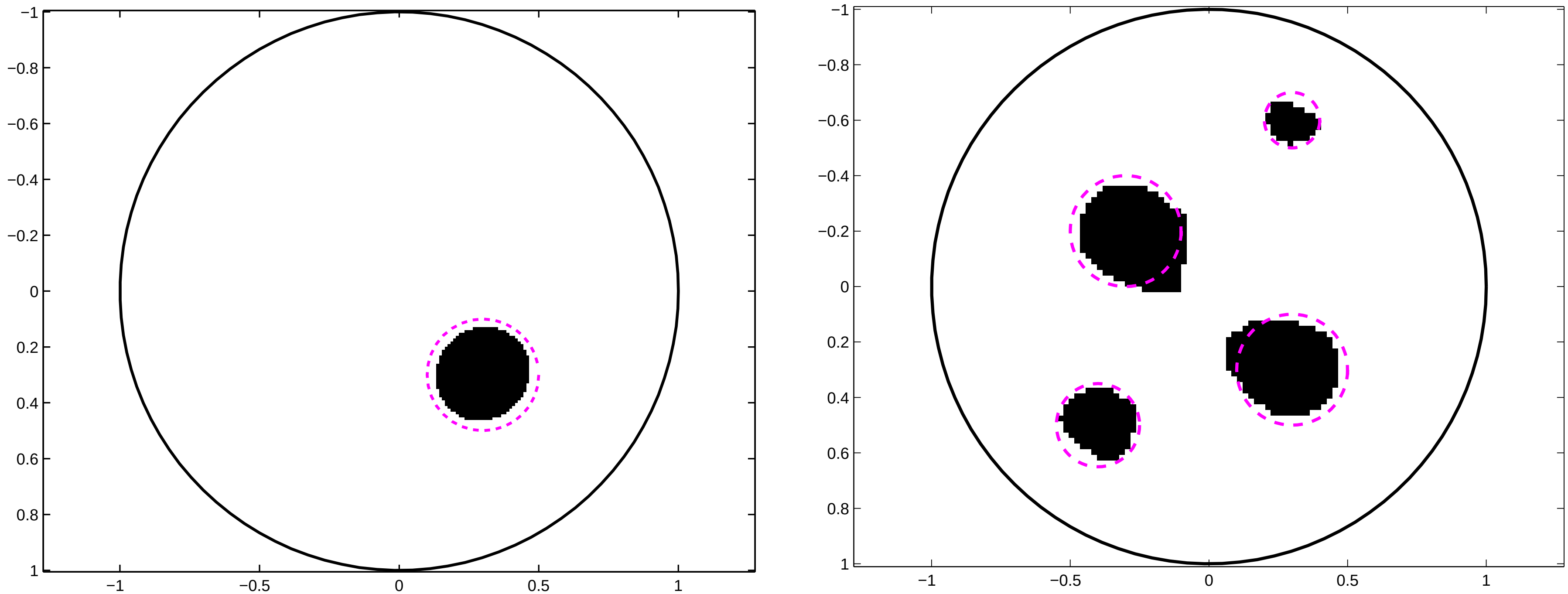
- Range test $v_z|_{\partial B} \in \mathcal{R}(\tilde{\Lambda}^{1/2})$ can be implemented numerically

  $\rightsquigarrow$ practical reconstruction algorithm.

- Ω does not have to be connected. No apriori information about the number of connected components is needed.

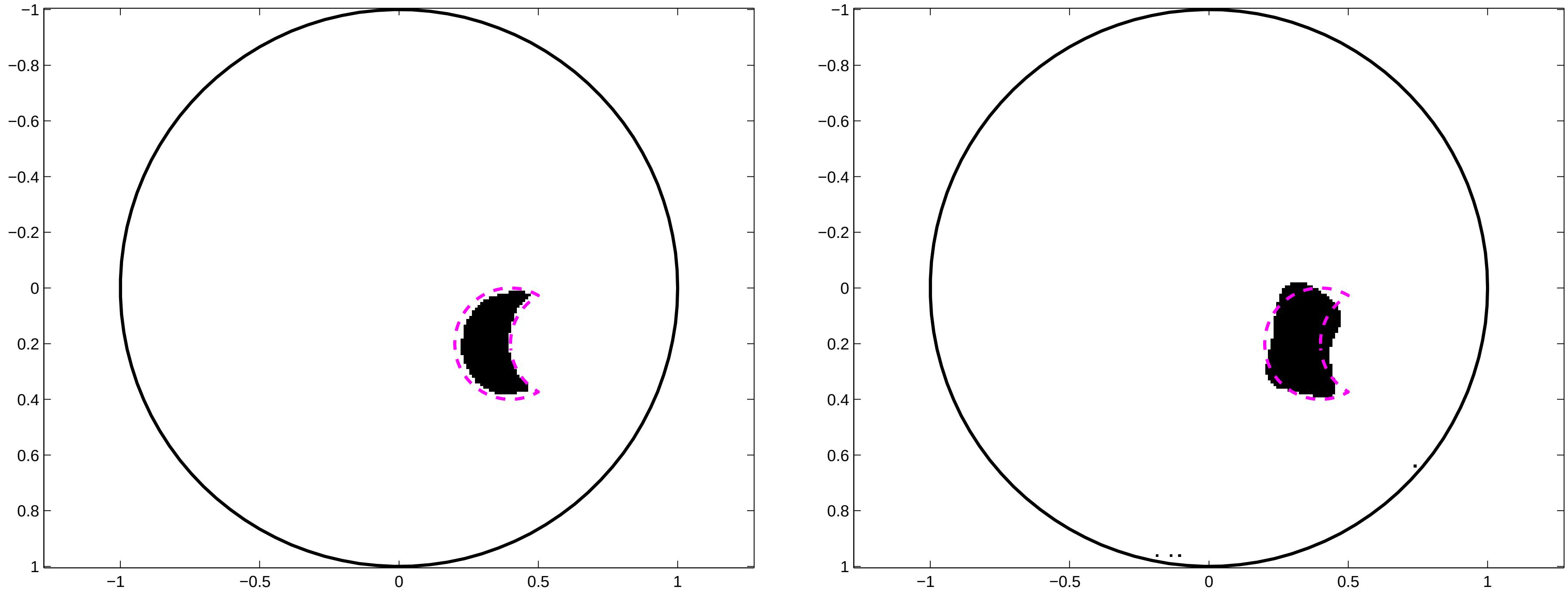
- Theory stays valid for partial boundary data, i. e. if heat flux is applied on a subset of ∂B and temperature is measured on the same part.

Numerical results



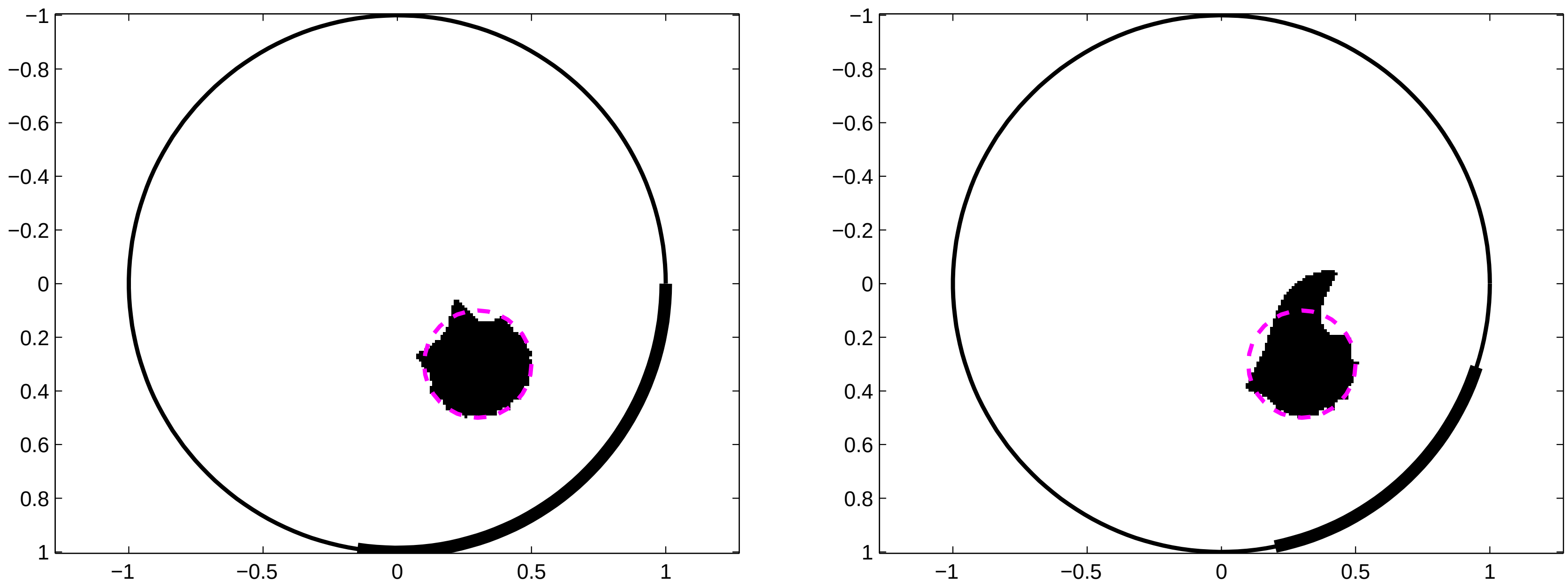
Reconstruction of a single and of multiple inclusions.

Numerical results



Reconstruction of a nonconvex inclusion
(left: no noise, right: 0.1% noise)

Numerical results



Reconstructions from partial boundary data