Electric potentials with localized divergence properties

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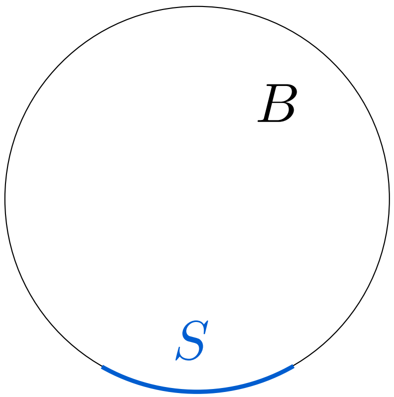
Overview

- Motivation and existence of localized potentials
- Construction and numerical examples
- Relation and consequences for the Factorization Method

Motivation and existence

Electrical impedance tomography



B:	bounded domain
$S \subseteq \partial B$:	relatively open subset
$\sigma \in L^\infty_+(B)$:	electrical conductivity in B
$g \in L^2_\diamond(S)$:	applied current on S

$\rightsquigarrow$ Electric potential $u \in H^1_\diamond(B)$ that solves

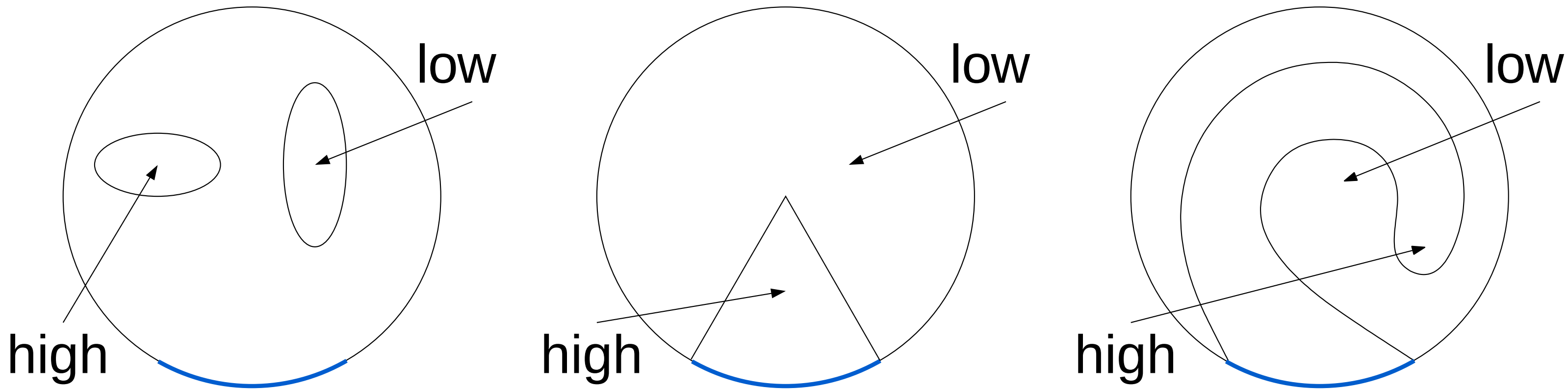
$$\nabla \cdot \sigma \nabla u = 0, \qquad \sigma \partial_\nu u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else.} \end{cases}$$

EIT: Measure $u|_S$ for one or several input currents g and reconstruct (properties of) σ from it.

Additional assumption: σ satisfies (UCP), i.e.,
if $u|_T = 0$, $\sigma \partial_\nu u|_T = 0$ for some open part T of a surface in $\overline{B}$ then $u = 0$.

Localized potentials

Can we localize (the energy of) the potentials in given subsets?



Restrictions:

- High energy parts have to be connected to the boundary.
- Because of (UCP) zero energy parts are not possible.

$\rightsquigarrow$ Goal: sequences (g_n) such that energy of (u_n) diverges on some subset while tending to zero on another.

Energy of a potential u: $\int_\Omega \sigma |\nabla u|^2 \, \mathrm{d}x \approx \int_\Omega |\nabla u|^2 \, \mathrm{d}x \approx \|u\|^2_{H^1_\diamond(\Omega)}$

Theoretical motivation

Calderon problem with local data:

Is σ uniquely determined by the full (local) current-to-voltage map

$$\Lambda_\sigma : \ L^2_\diamond(S) \to L^2_\diamond(S), \quad g \mapsto u|_S \text{ ?}$$

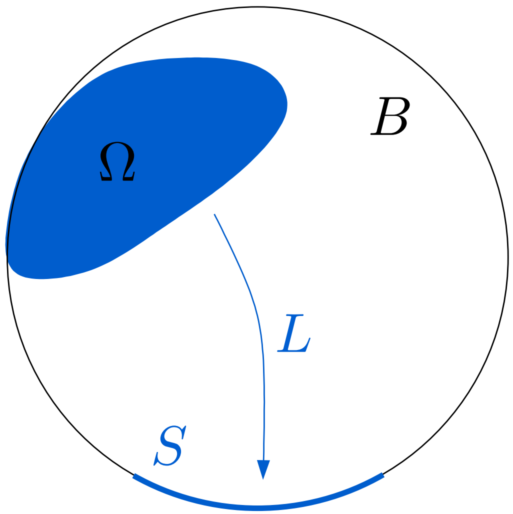
Monotonicity property:

$$\int_B (\sigma_1 - \sigma_2)|\nabla u_2|^2 \,\mathrm{d}x \geq \langle (\Lambda_{\sigma_2} - \Lambda_{\sigma_1})g, g\rangle \geq \int_B (\sigma_1 - \sigma_2)|\nabla u_1|^2 \,\mathrm{d}x$$

$\rightsquigarrow$ If $\sigma_1 - \sigma_2 > 0$ in some region where we can localize the electric energy $|\nabla u_1|^2$ then $\Lambda_{\sigma_1} \neq \Lambda_{\sigma_2}$.

"A higher conductivity in such a region can not be balanced out."

Virtual Measurements



$f \in (H^1_\diamond(\Omega))'$: applied source on Ω

$$L_\Omega : \ (H^1_\diamond(\Omega))' \to L^2_\diamond(S), \quad f \mapsto u|_S,$$

where $u \in H^1_\diamond(B)$ solves

$$\int_B \sigma \nabla u \cdot \nabla v \,\mathrm{d}x = \langle f, v|_\Omega \rangle \quad \text{for all } v \in H^1_\diamond(B).$$

If $\overline{\Omega} \subset B$: $\quad \nabla \cdot \sigma \nabla u = f, \quad \sigma \partial_\nu u|_{\partial B} = 0.$

(UCP) yields: If $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$, $B \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ is connected and its boundary contains S then $\mathcal{R}(L_{\Omega_1}) \cap \mathcal{R}(L_{\Omega_2}) = 0$.

Dual operator $L'_\Omega : \ L^2_\diamond(S) \to H^1_\diamond(\Omega), \quad g \mapsto u|_\Omega$, where u solves

$$\nabla \cdot \sigma \nabla u = 0, \qquad \sigma \partial_\nu u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else.} \end{cases}$$

Some functional analysis

Lemma

Let X, Y be two reflexive Banach spaces, $A \in \mathcal{L}(X,Y)$, $y \in Y$. Then

$$y \in \mathcal{R}(A) \quad \text{iff} \quad |\langle y', y\rangle| \leq C \, \|A' y'\| \quad \forall y' \in Y'.$$

Corollary

If $\|L'_{\Omega_1} g\| \leq C \, \|L'_{\Omega_2} g\|$ for all applied currents g, i.e., $\|u|_{\Omega_1}\| \leq C \, \|u|_{\Omega_2}\|$ for the corresponding potentials u, then $\mathcal{R}(L_{\Omega_1}) \subseteq \mathcal{R}(L_{\Omega_2})$.

Contraposition

If $\mathcal{R}(L_{\Omega_1}) \not\subseteq \mathcal{R}(L_{\Omega_2})$ then there exist currents (g_n) such that the corresponding potentials (u_n) satisfy

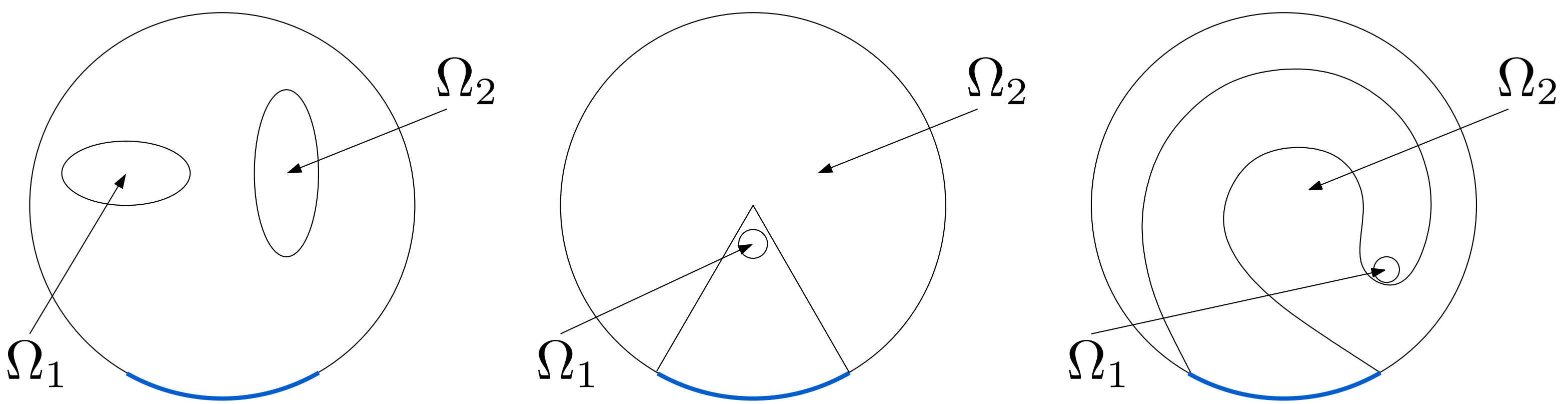
$$\|u_n|_{\Omega_1}\|_{H^1_\diamond(\Omega_1)} \to \infty \quad \text{and} \quad \|u_n|_{\Omega_2}\|_{H^1_\diamond(\Omega_2)} \to 0.$$

Existence of localized potentials

Theorem

If $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$, $B \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ is connected and its boundary contains S, then there exists currents (g_n) such that (the energy of) the corresponding potentials (u_n) diverges on Ω_1 while tending to zero on Ω_2, i.e.,

$$\int_{\Omega_1} \sigma |\nabla u_n|^2 \,\mathrm{d}x \to \infty, \quad \text{and} \quad \int_{\Omega_2} \sigma |\nabla u_n|^2 \,\mathrm{d}x \to 0.$$



Result uses only ellipticity properties, thus also holds e.g. for linear elasticity, electro- and magnetostatics.

Theoretical consequence

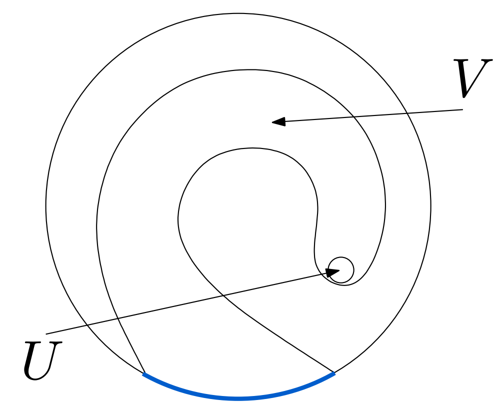
Corollary

Let $\sigma_1, \sigma_2 \in L^\infty_+(B)$ satisfy (UCP) and Λ_{σ_1}, Λ_{σ_2} be the corresponding current-to-volage-maps.

If $\sigma_2 \geq \sigma_1$ in some neighbourhood V of S and $\sigma_2 - \sigma_1 \in L^\infty_+(U)$ for some open $U \subseteq V$ then there exists (g_n) such that

$$\langle (\Lambda_{\sigma_2} - \Lambda_{\sigma_1}) g_n, g_n \rangle \to \infty,$$

so in particular $\Lambda_{\sigma_2} \neq \Lambda_{\sigma_1}$.



Consequence:

Piecewise constant conductivities σ are uniquely determined by Λ_σ (cf. e.g. Druskin, 1998, for local boundary data for of a halfspace)

Construction

Construction

More constructive version of the functional analysis:

Let $h \in \mathcal{R}(L_{\Omega_1})$, $h \notin \mathcal{R}(L_{\Omega_2})$. Define $\gamma_\alpha \in L^2_\diamond(S)$ by

$$\gamma_\alpha = (L_{\Omega_2} L_{\Omega_2}^* + \alpha I)^{-1} h, \qquad \alpha > 0.$$

Then

$$\|L'_{\Omega_2}\gamma_\alpha\|^2 \leq C \|L'_{\Omega_1}\gamma_\alpha\| \quad \text{and} \quad \|L'_{\Omega_2}\gamma_\alpha\| \to \infty \quad \text{for } \alpha \to 0.$$

So for the currents $g_\alpha := \dfrac{1}{\|L'_{\Omega_2}\gamma_\alpha\|^{3/2}}\gamma_\alpha$, the corresponding potentials u_α satisfy

$$\|L'_{\Omega_1} g_\alpha\|^2 \approx \int_{\Omega_1} \sigma |\nabla u_\alpha|^2 \, \mathrm{d}x \to \infty,$$

$$\|L'_{\Omega_2} g_\alpha\|^2 \approx \int_{\Omega_2} \sigma |\nabla u_\alpha|^2 \, \mathrm{d}x \to 0.$$

Construction

Even more specific for $\sigma = 1$:

- h_z: boundary data of a electric dipole in $z \in B$, i.e., $h_z = u_z|_S$, where

$$\Delta u_z = d \cdot \nabla \delta_z, \qquad \partial_\nu u_z|_{\partial B} = 0$$

  ($d \in \mathbb{R}^n$, $|d| = 1$ fixed arbitrary direction).

- If $B \setminus \overline{\Omega}$ is connected and its boundary contains S, then for $z \notin \partial\Omega$:

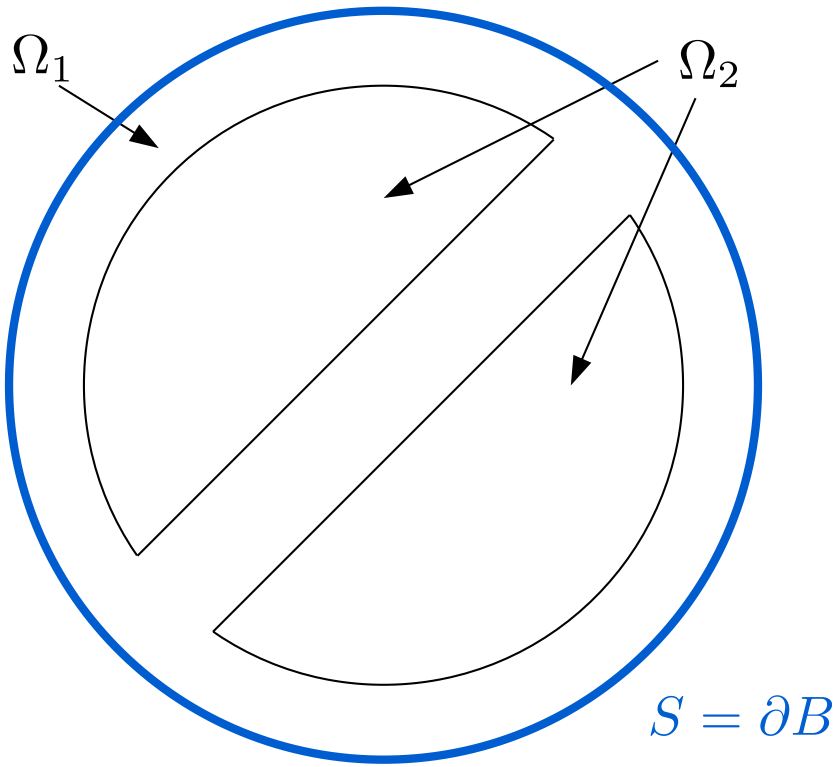
$$h_z \in \mathcal{R}(L_\Omega) \quad \text{iff} \quad z \in \Omega.$$

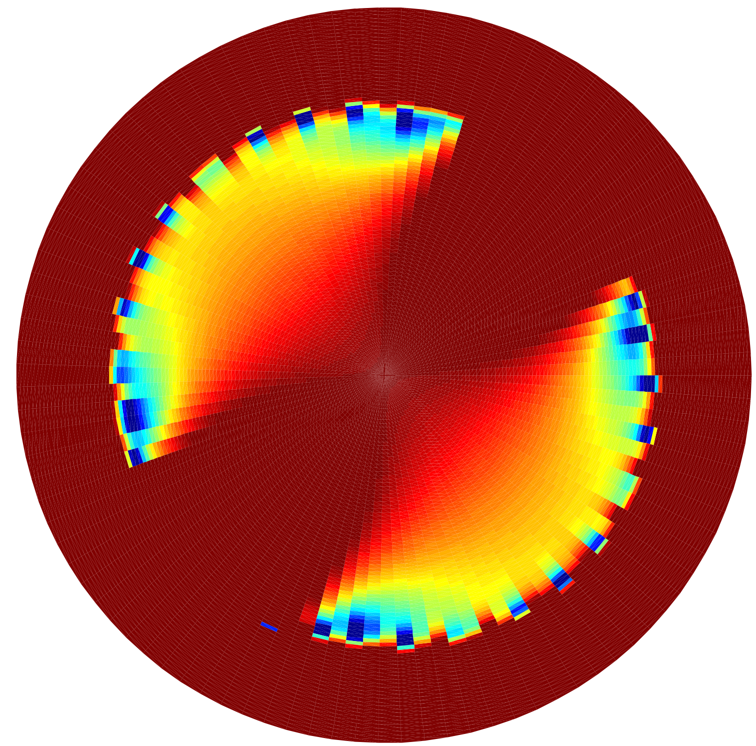
- Define currents $g_{\alpha,z}$, potentials $u_{\alpha,z}$ as on the last slide, then

$$\int_{\Omega_1} \sigma |\nabla u_{\alpha,z}|^2 \,\mathrm{d}x \to \infty, \qquad \int_{\Omega_2} \sigma |\nabla u_{\alpha,z}|^2 \,\mathrm{d}x \to 0$$

  for every neighbourhood Ω_1 of z.

Numerical example





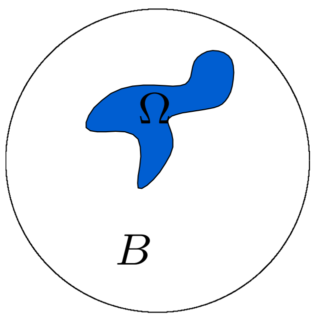
$z = (0, 0)$, color axis scaled logarithmically and cropped.

$\rightsquigarrow$ *Electric potentials can be localized in very general domains but the problem is very ill-posed.*

Factorization Method

Factorization Method

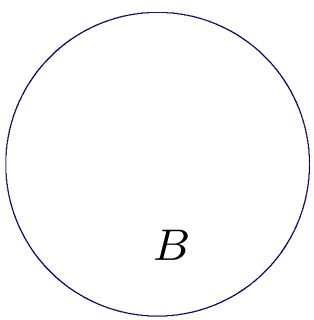
Special case of EIT: locate inclusions in known background medium.



Current-to-voltage map with inclusion:

$$\Lambda_1 : \; g \mapsto u_1|_{\partial B},$$

where u_1 solves $\nabla \cdot \sigma \nabla u_1 = 0$, $\partial_\nu u_1|_{\partial B} = g$
with $\sigma = 1 + \sigma_1 \chi_\Omega$, $\sigma_1 > 0$.



Current-to-voltage map without inclusion:

$$\Lambda_0 : \; g \mapsto u_0|_{\partial B},$$

where u_0 solves $\Delta u_0 = 0$, $\partial_\nu u_0|_{\partial B} = g$
with $\sigma = 1$.

Factorization method reconstructs Ω from $\Lambda := \Lambda_0 - \Lambda_1$
by using that for $z \notin \partial\Omega$:

$$h_z \in \mathcal{R}(\Lambda^{1/2}) \quad \text{iff} \quad z \in \Omega.$$

Factorization Method

Factorization Method: for $z \notin \partial\Omega$

$$z \in \Omega \quad \text{iff} \quad h_z \in \mathcal{R}(\Lambda^{1/2}).$$

- Numerical implementations usually calculate regularized preimage

  $$\psi_{z,\alpha} \approx \Lambda^{-1/2} h_z$$

  and use that (for $z \notin \partial\Omega$)

  $$z \in \Omega \quad \text{iff} \quad \|\psi_{z,\alpha}\| \text{ bounded.}$$

  $\psi_{z,\alpha} \approx \Lambda^{-1/2} h_z$ *has no physical interpretation.*

- Similar criterion (for $z \notin \partial\Omega$) uses $g_{z,\alpha} := \Lambda^*(\Lambda\Lambda^* + \alpha I)^{-1} h_z \approx \Lambda^{-1} h_z$

  $$z \in \Omega \quad \text{iff} \quad \|L'_\Omega g_{z,\alpha}\| \text{ bounded.}$$

  Physical interpretation: $L'_\Omega g_{z,\alpha} = u_{z,\alpha}|_\Omega$ with electric potential $u_{z,\alpha}$ generated by $g_{z,\alpha}$ (in homogeneous body).

Factorization Method

Lemma

There exist $C, C' > 0$:

$$|\nabla u_{z,\alpha}(z)| \leq C \; \|L_\Omega g_{z,\alpha}\| \qquad \text{for } z \in \Omega,$$

$$|\nabla u_{z,\alpha}(z)| \geq C' \, \|L_\Omega g_{z,\alpha}\|^2 \qquad \text{for } z \notin \Omega.$$

Variant of the Factorization Method (for $z \notin \partial\Omega$):

$$z \in \Omega \quad \text{iff} \quad |\nabla u_{z,\alpha}(z)| \text{ bounded.}$$

(EIT-analogue of Arens' variant of the FM / LSM for inverse scattering).

Interpretation in context of localized potentials:

For fixed $z \notin \Omega$ (and after appropriate scaling) $u_{z,\alpha}(\cdot)$ are potentials with

- energy tending to zero on Ω,
- energy tending to infinity on every neighbourhood of z.

Summary

- In theory, electric potentials can be localized on quite arbitrary regions as long as they are connected to the boundary.
- In practice, such potentials can be calculated by solving ill-posed equations.
- The approximated preimages of the Factorization Method (after appropriate scaling) can be interpreted as potentials that localize outside the unknown inclusions.