Localized potentials in electrical impedance tomography

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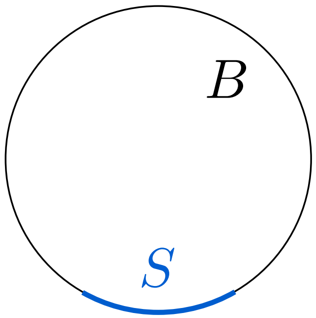
Overview

- Motivation
- Existence of localized potentials
- Construction and numerical examples
- Detecting inclusions in EIT

Motivation

Electrical impedance tomography



B: bounded domain

$S \subseteq \partial B$: relatively open subset

$\sigma \in L^\infty_+(B)$: electrical conductivity in B

$g \in L^2_\diamond(S)$: applied current on S

$\rightsquigarrow$ Electric potential $u \in H^1_\diamond(B)$ that solves

$$\nabla \cdot \sigma \nabla u = 0, \qquad \sigma \partial_\nu u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else.} \end{cases}$$

EIT: Measure $u|_S$ for one or several input currents g and reconstruct (properties of) σ from it.

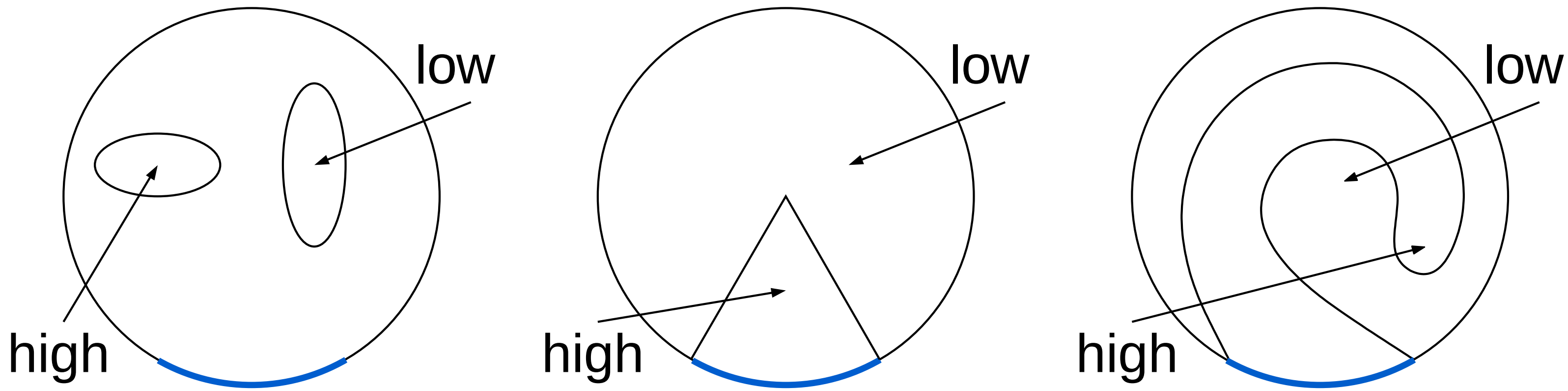
Regularity assumption:

σ satisfies (UCP) in connected neighbourhoods U of S, i.e.,

$$\nabla \cdot \sigma \nabla u = 0 \text{ in } U, \quad \begin{cases} u|_S = 0,\ \sigma \partial_\nu u|_S = 0 & \Longrightarrow \quad u = 0. \\ u|_V = \text{const.}, V \subset U \text{ open} & \Longrightarrow \quad u = \text{const.} \end{cases}$$

Localized potentials

Can we localize (the energy of) the potentials in given subsets?



Restrictions:

- High energy parts have to be connected to the boundary.
- Because of (UCP) zero energy parts are not possible.

$\rightsquigarrow$ Goal: sequences (g_n) such that energy of (u_n) diverges on some subset while tending to zero on another.

Energy of a potential u: $\int_\Omega \sigma |\nabla u|^2 \, \mathrm{d}x \approx \int_\Omega |\nabla u|^2 \, \mathrm{d}x \approx \|u\|^2_{H^1_\diamond(\Omega)}$

Theoretical motivation

Calderon problem with partial data:

Is σ uniquely determined by the (local) current-to-voltage map

$$\Lambda_\sigma : L^2_\diamond(S) \to L^2_\diamond(S), \quad g \mapsto u|_S \text{ ?}$$

For measurements on whole boundary $S = \partial B$:

- Identifiabilty question posed by Calderon 1980.
- For smooth σ answered positively by Sylvester and Uhlmann 1987 for $n \geq 3$ and by Nachmann 1996 for $n = 2$.
- For $n = 2$ and general $\sigma \in L^\infty_+$ answered positively by Astala and Päivärinta 2006.
- Still an open question for general $\sigma \in L^\infty_+$ (with or without (UCP)) for $n \geq 3$.

Theoretical motivation

Connection between Calderon problem (with $S \subseteq \partial B$) and localized potentials:

- Monotonicity property:

  Let u_1, u_2 be electric potentials for conductivities σ_1, σ_2 created by the same boundary current $g \in L^2_\diamond(S)$. Then

$$\int_B (\sigma_1 - \sigma_2)|\nabla u_2|^2 \,\mathrm{d}x \geq ((\Lambda_{\sigma_2} - \Lambda_{\sigma_1})g, g) \geq \int_B (\sigma_1 - \sigma_2)|\nabla u_1|^2 \,\mathrm{d}x.$$

$\rightsquigarrow$ If $\sigma_1 - \sigma_2 > 0$ in some region where we can localize the electric energy $|\nabla u_1|^2$ then $\Lambda_{\sigma_1} \neq \Lambda_{\sigma_2}$.

"A higher conductivity in such a region can not be balanced out."

Known results on loc. potentials

- Potential can be concentrated around $z \in S$ if $\sigma \in C^2$ around z.
  (Kohn, Vogelius 1984)

$\rightsquigarrow$ $\sigma|_S$ and its derivatives on S are determined by local voltage-to-current map.

- Using Runge's approximation property the high energy part can be "shifted" into the interior of B.
  (Kohn, Vogelius 1985)

$\rightsquigarrow$ Piecewise analytic σ is determined by local voltage-to-current map.

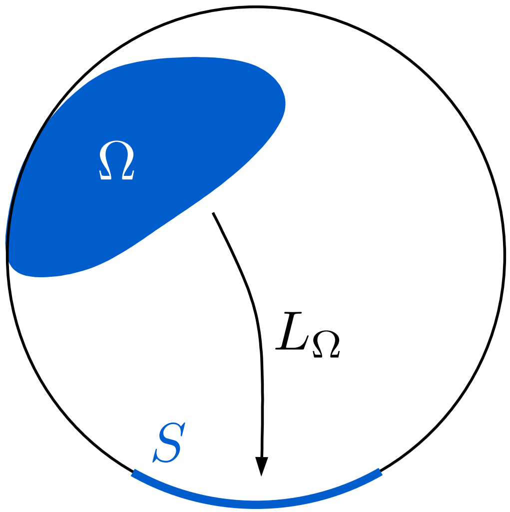
- In dimension $n \geq 3$, C^2-conductivities σ are determined by the local voltage-to-current map, .
  (Kenig, Sjöstrand, Uhlmann 2007,
  uses the Kohn-Vogelius result for determining $\sigma|_S$, $\partial_\nu \sigma|_S$.)

In this talk:

- Localized potentials exist for arbitrary $\sigma \in L^\infty_+$ that fulfill (UCP).

Existence of localized potentials

Virtual Measurements



$f \in (H^1_\diamond(\Omega))'$: applied source on Ω

$$L_\Omega : \ (H^1_\diamond(\Omega))' \to L^2_\diamond(S), \quad f \mapsto u|_S,$$

where $u \in H^1_\diamond(B)$ solves

$$\int_B \sigma \nabla u \cdot \nabla v \,\mathrm{d}x = \langle f, v|_\Omega \rangle \quad \text{for all } v \in H^1_\diamond(B).$$

If $\overline{\Omega} \subset B$: $\quad \nabla \cdot \sigma \nabla u = f, \quad \sigma \partial_\nu u|_{\partial B} = 0.$

(UCP) yields: If $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$, $B \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ is connected and its boundary contains S then $\mathcal{R}(L_{\Omega_1}) \cap \mathcal{R}(L_{\Omega_2}) = 0$.

Dual operator $L'_\Omega : \ L^2_\diamond(S) \to H^1_\diamond(\Omega), \quad g \mapsto u|_\Omega$, where u solves

$$\nabla \cdot \sigma \nabla u = 0, \qquad \sigma \partial_\nu u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else.} \end{cases}$$

Some functional analysis

Lemma

Let X, Y be two reflexive Banach spaces, $A \in \mathcal{L}(X, Y)$, $y \in Y$. Then

$$y \in \mathcal{R}(A) \quad \text{iff} \quad |\langle y', y \rangle| \leq C \, \|A' y'\| \quad \forall y' \in Y'.$$

Corollary

If $\|L'_{\Omega_1} g\| \leq C \, \|L'_{\Omega_2} g\|$ for all applied currents g, i.e., $\|u|_{\Omega_1}\| \leq C \, \|u|_{\Omega_2}\|$ for the corresponding potentials u, then $\mathcal{R}(L_{\Omega_1}) \subseteq \mathcal{R}(L_{\Omega_2})$.

Contraposition

If $\mathcal{R}(L_{\Omega_1}) \not\subseteq \mathcal{R}(L_{\Omega_2})$ then there exist currents (g_n) such that the corresponding potentials (u_n) satisfy

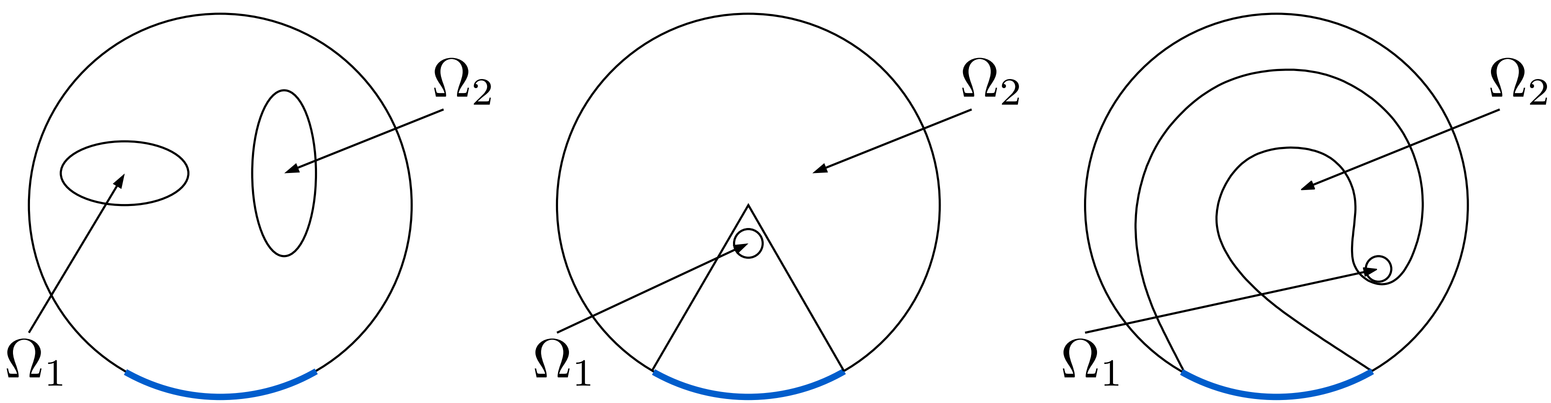
$$\|u_n|_{\Omega_1}\|_{H^1_\diamond(\Omega_1)} \to \infty \quad \text{and} \quad \|u_n|_{\Omega_2}\|_{H^1_\diamond(\Omega_2)} \to 0.$$

Existence of localized potentials

Theorem

If $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$, $B \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ is connected and its boundary contains S, then there exists currents (g_n) such that (the energy of) the corresponding potentials (u_n) diverges on Ω_1 while tending to zero on Ω_2, i.e.,

$$\int_{\Omega_1} \sigma |\nabla u_n|^2 \, \mathrm{d}x \to \infty, \quad \text{and} \quad \int_{\Omega_2} \sigma |\nabla u_n|^2 \, \mathrm{d}x \to 0.$$



Result uses only ellipticity properties, thus also holds e.g. for linear elasticity, electro- and magnetostatics.

Theoretical consequence

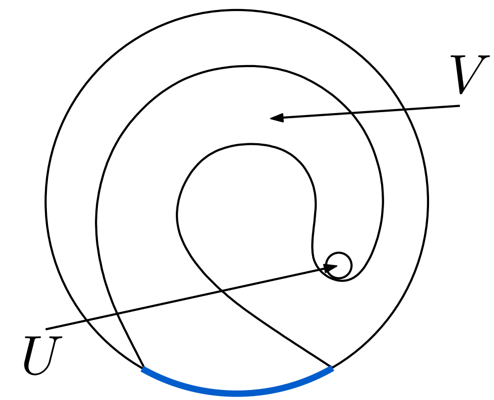
Corollary

Let $\sigma_1, \sigma_2 \in L^\infty_+(B)$ satisfy (UCP) and Λ_{σ_1}, Λ_{σ_2} be the corresponding current-to-volage-maps.

If $\sigma_2 \geq \sigma_1$ in some neighbourhood V of S and $\sigma_2 - \sigma_1 \in L^\infty_+(U)$ for some open $U \subseteq V$ then there exists (g_n) such that

$$\langle (\Lambda_{\sigma_2} - \Lambda_{\sigma_1}) g_n, g_n \rangle \to \infty,$$

so in particular $\Lambda_{\sigma_2} \neq \Lambda_{\sigma_1}$.



Consequences (already known from the Kohn-Vogelius result):

- $\sigma|_S$ and its derivatives on S are uniquely determined by Λ_σ.
- Piecewise analytic conductivities σ are uniquely determined by Λ_σ.

Construction

Construction

More constructive version of the functional analysis:

Let $h \in \mathcal{R}(L_{\Omega_1})$, $h \notin \mathcal{R}(L_{\Omega_2})$. Define $\gamma_\alpha \in L^2_\diamond(S)$ by

$$\gamma_\alpha = (L_{\Omega_2} L_{\Omega_2}^* + \alpha I)^{-1} h, \qquad \alpha > 0.$$

Then

$$\|L'_{\Omega_2}\gamma_\alpha\|^2 \leq C \|L'_{\Omega_1}\gamma_\alpha\| \quad \text{and} \quad \|L'_{\Omega_2}\gamma_\alpha\| \to \infty \quad \text{for } \alpha \to 0.$$

So for the currents $g_\alpha := \dfrac{1}{\|L'_{\Omega_2}\gamma_\alpha\|^{3/2}}\gamma_\alpha$, the corresponding potentials u_α satisfy

$$\|L'_{\Omega_1} g_\alpha\|^2 \approx \int_{\Omega_1} \sigma |\nabla u_\alpha|^2 \, \mathrm{d}x \to \infty,$$

$$\|L'_{\Omega_2} g_\alpha\|^2 \approx \int_{\Omega_2} \sigma |\nabla u_\alpha|^2 \, \mathrm{d}x \to 0.$$

Construction

Even more specific for $\sigma = 1$:

- h_z: boundary data of a electric dipole in $z \in B$, i.e., $h_z = u_z|_S$, where

$$\Delta u_z = d \cdot \nabla \delta_z, \qquad \partial_\nu u_z|_{\partial B} = 0$$

($d \in \mathbb{R}^n$, $|d| = 1$ fixed arbitrary direction).

- If $B \setminus \overline{\Omega}$ is connected and its boundary contains S, then for $z \notin \partial\Omega$:

$$h_z \in \mathcal{R}(L_\Omega) \quad \text{iff} \quad z \in \Omega.$$

- Define currents $g_{\alpha,z}$, potentials $u_{\alpha,z}$ as on the last slide, then

$$\int_{\Omega_1} \sigma |\nabla u_{\alpha,z}|^2 \,\mathrm{d}x \to \infty, \qquad \int_{\Omega_2} \sigma |\nabla u_{\alpha,z}|^2 \,\mathrm{d}x \to 0$$

for every neighbourhood Ω_1 of z.

Implementation

Implementation of $L_\Omega L_\Omega^*$.

$$L_\Omega L_\Omega^* : \; L^2_\diamond(S) \to L^2_\diamond(S), \quad g \mapsto u|_S,$$

where $u \in H^1_\diamond(B)$ solves (for $\overline{\Omega} \subset B$)

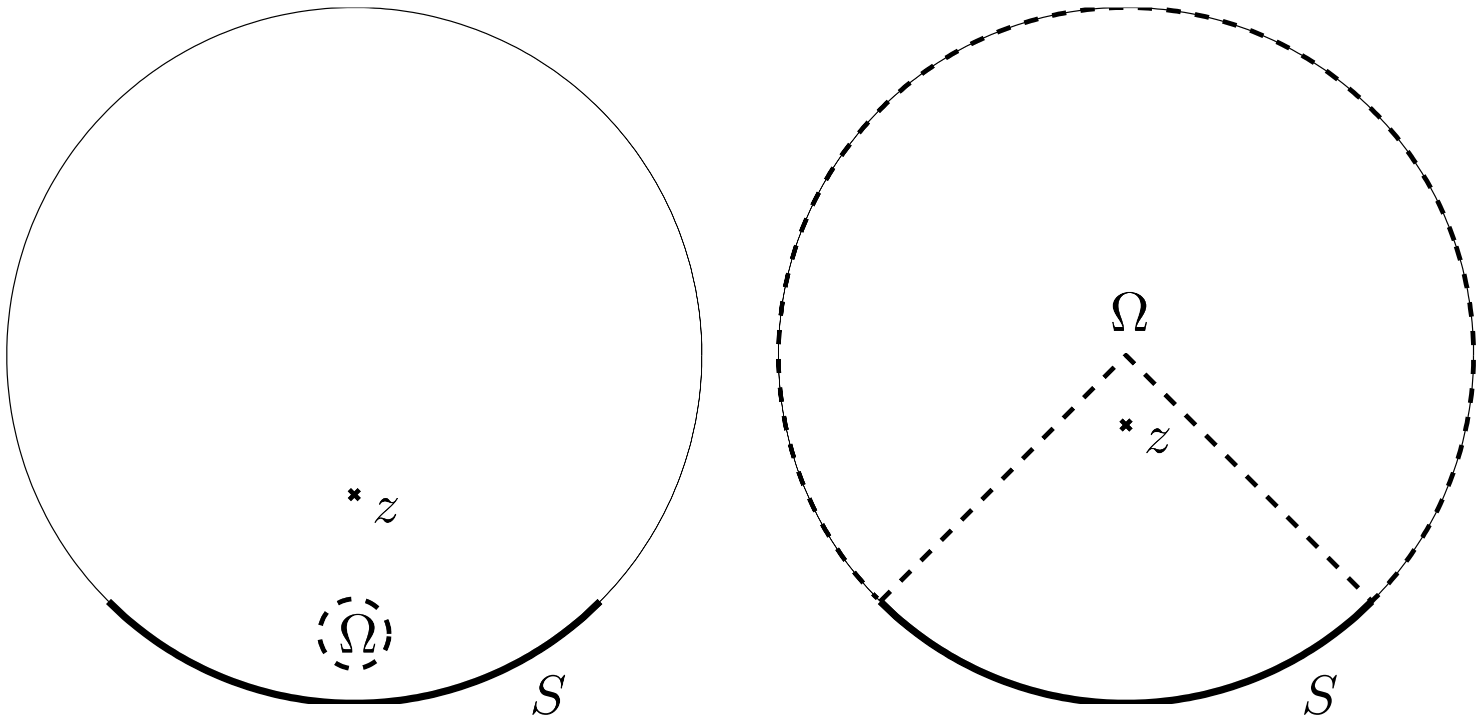
$$\nabla \cdot \sigma \nabla u = \nabla \cdot \chi_\Omega \nabla v \quad \text{and} \quad \sigma \partial_\nu u|_{\partial B} = 0,$$

with the solution $v \in H^1_\diamond(B)$ of

$$\nabla \cdot \sigma \nabla v = 0 \quad \text{and} \quad \sigma \partial_\nu v|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{on } \partial B \setminus S. \end{cases}$$

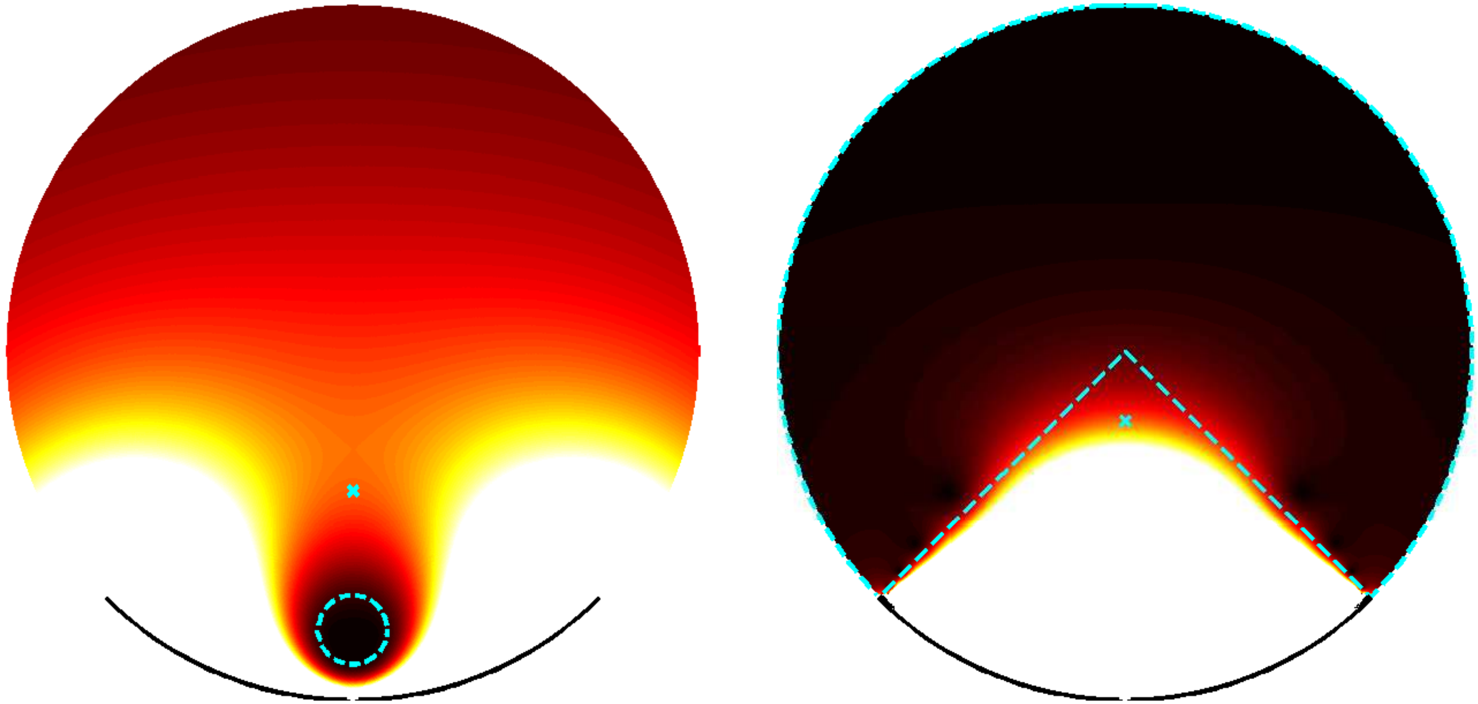
v and u are easily computed using standard solvers for linear elliptic equations (here: Comsol).

Numerical examples



Find a potential with high energy around z and low energy in Ω!

Numerical examples



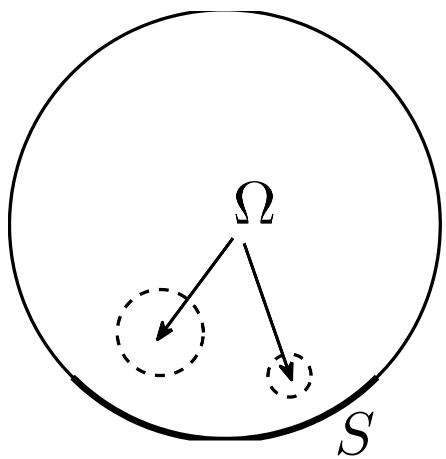
Plots of $|\nabla u_{\alpha,z}|$, α choosen "by hand", color axis cropped above $2|\nabla u_{\alpha,z}(z)|$.

$\rightsquigarrow$ *Electric potentials can be localized in very general domains but the problem is very ill-posed.*

Detecting inclusions in EIT

Detecting inclusions in EIT

Special case of EIT: locate inclusions in known background medium.



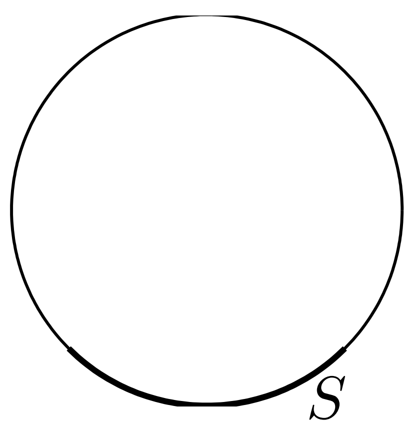
Current-to-voltage map with inclusion:

$$\Lambda_1 : \; g \mapsto u_1|_{\partial B},$$

where u_1 solves

$$\nabla \cdot \sigma \nabla u_1 = 0 \quad \partial_\nu u_1|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else}, \end{cases}$$

with $\sigma = 1 + \sigma_1 \chi_\Omega$, $\sigma_1 > 0$.



Current-to-voltage map without inclusion:

$$\Lambda_0 : \; g \mapsto u_0|_{\partial B},$$

where u_0 solves the analogous equation with $\sigma = 1$.

Goal: Reconstruct Ω from comparing Λ_1 with Λ_0.

Detecting inclusions in EIT

"Straight-forward" detection algorithm using localized potentials:

- Construct current g leading to localized potential that is large on some test domain U and small outside.
- Monotonicity property

  $\rightsquigarrow$ $((\Lambda_0 - \Lambda_1)g, g)$ large if U intersects Ω.

  $\rightsquigarrow$ Ω can be found by "slowly enlarging appropriately chosen U".

However,

- This needs a large number of localized potentials.
- Analogous idea can be realized much easier using other special singular potentials (Ikehata's probe method, Potthast's point source/singular sources method).

Last few slides: "less straight-forward" (yet very simple) detection algo.

Virtual measurements again

Connection between $\Lambda_0 - \Lambda_1$ and virtual measurements L_Ω:

Lemma

There exist $c, C > 0$ such that

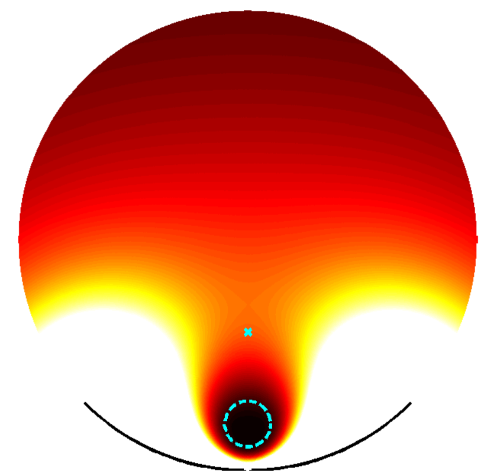
$$c \left\| L_\Omega^* g \right\|^2 \leq ((\Lambda_0 - \Lambda_1) g, g) \leq C \left\| L_\Omega^* g \right\|^2 \quad \text{for all } g \in L^2_\diamond(S),$$

so, roughly speaking, $L_\Omega L_\Omega^* \approx \Lambda_0 - \Lambda_1$.

Only $L_\Omega L_\Omega^*$ is needed to construct a localized potential that is large in some $z \notin \Omega$ and small in Ω.

$\rightsquigarrow$ Simple reconstruction algorithm:

Given a $z \notin \overline{\Omega}$ (must be known!), use $\Lambda := \Lambda_0 - \Lambda_1$ to create such a potential and locate Ω from it.



Reconstruction algorithm

Theorem

$z \notin \overline{\Omega}$, h_z: electric dipole in z, $g_z^\alpha := \Lambda^*(\Lambda\Lambda^* + \alpha I)^{-1} h_z \approx \Lambda^{-1} h_z$,
u_z^α hom. potential for currents $g_z^\alpha/(\Lambda g_z^\alpha, g_z^\alpha)^{3/2}$. Then

$$|\nabla u_z^\alpha(z)| \to \infty \quad \text{and} \quad |\nabla u_z^\alpha(x)| \to 0 \quad \text{for } x \in \Omega.$$

Connection to the Factorization Method *(slightly simplified)*:

- Factorization Method (Kirsch, Hanke, Brühl,...):
  $$z \notin \Omega \text{ if and only if } \|\Lambda^{-1/2} h_z\| \to \infty.$$
- EIT-analogue of Arens' variant of this criterion:
  $$z \notin \Omega \text{ if and only if } |\nabla v_z^\alpha(z)| \to \infty,$$
  where v_z^α hom. potential for currents $g_z^\alpha \approx \Lambda^{-1} h_z$.
- Here: $z \notin \overline{\Omega}$ fixed. $|\nabla u_z^\alpha(x)| \to 0$ if $x \in \Omega$

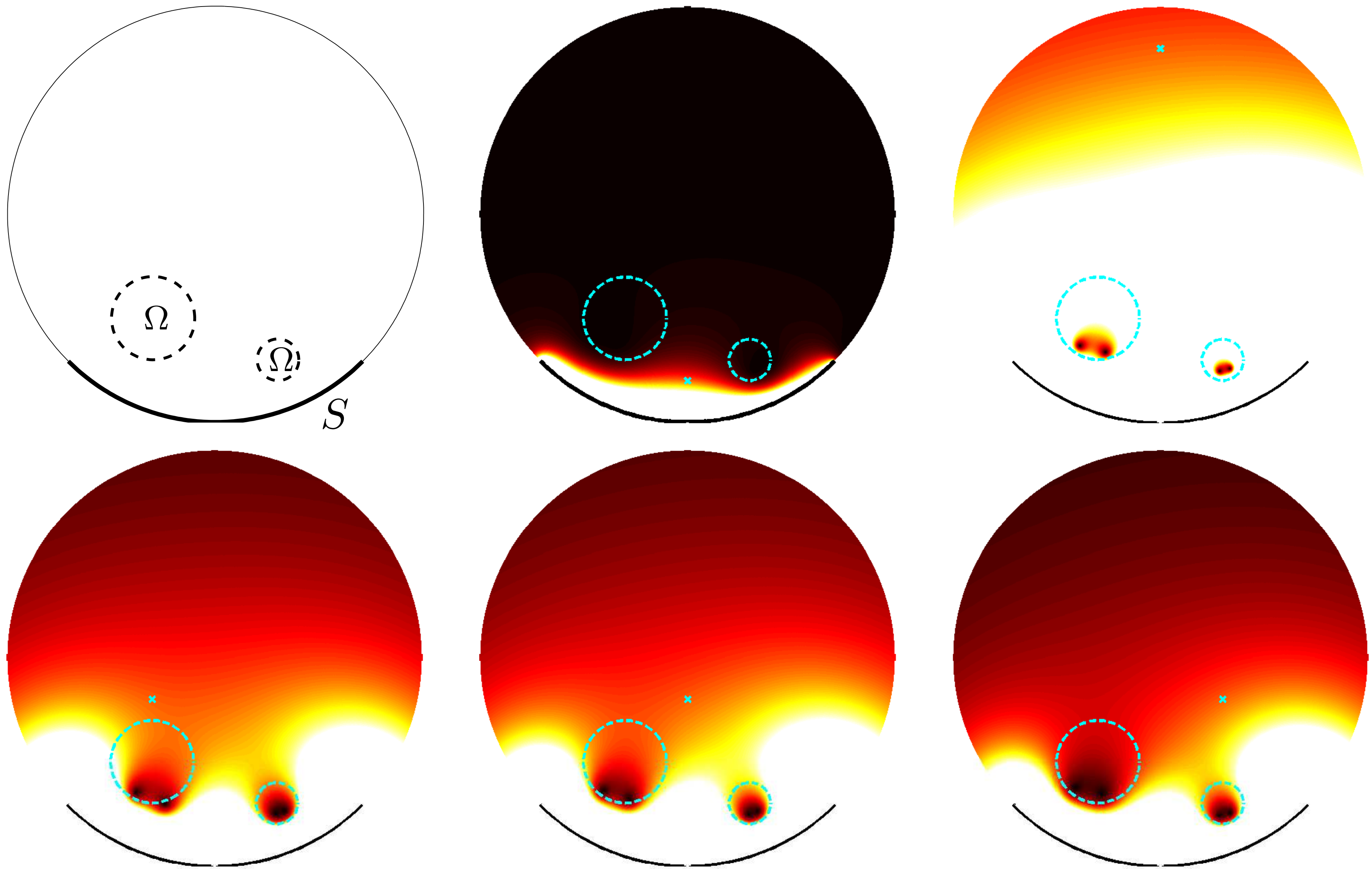
Properties

Reconstruction algorithm: Given $\Lambda = \Lambda_0 - \Lambda_1$, $z \notin \overline{\Omega}$,
calculate potential u_z^α and "mark all points x where $|\nabla u_z^\alpha(x)|$ is small."

- Needs to invert measurement matrix Λ for only one right hand side.
  (Can be done iteratively so that only few measurements are needed.)
- Needs to solve only one homogeneous forward problem.
- Independent of number of inclusions or jump amplitude.
- $|\nabla u_z^\alpha|$ may be small outside Ω. In theory, method finds only a superset!
- In practice, how can small values be distinguished from large values?

Method delivers a very quick, rough reconstruction of the inlusions.

Numerical example



Summary

- In theory, localized electric potentials exist for almost arbitrary conductivities and on almost arbitrary regions as long as they are connected to the boundary.
- Consequence for the Calderon problem for partial data:

  Two conductivities can be distinguished if one is larger in some part that is connected to the boundary.
- In practice, localized potentials can be calculated by solving ill-posed equations.
- For detecting inclusions in EIT, a quick rough reconstruction can be obtained by calculating a localized potential.