Electric potentials with localized divergence properties

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- Motivation and existence of localized potentials
- Construction and numerical examples
- Relation and consequences for the Factorization Method



Motivation and existence



Electrical impedance tomography



 $\begin{array}{lll} B: & \text{bounded domain} \\ S \subseteq \partial B: & \text{relatively open subset} \\ \sigma \in L^\infty_+(B): & \text{electrical conductivity in } B \\ g \in L^2_\diamond(S): & \text{applied current on } S \end{array}$

 \rightsquigarrow Electric potential $u \in H^1_{\diamond}(B)$ that solves

$$abla \cdot \sigma \nabla u = 0, \qquad \sigma \partial_{\nu} u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else.} \end{cases}$$

EIT: Measure $u|_S$ for one or several input currents g and reconstruct (properties of) σ from it.

Additional assumption: σ satisfies (UCP), i.e.,

if $u|_T = 0$, $\sigma \partial_{\nu} u|_T = 0$ for some open part T of a surface in \overline{B} then u = 0.





Localized potentials

Can we localize (the energy of) the potentials in given subsets?



Restrictions:

- High energy parts have to be connected to the boundary.
- Because of (UCP) zero energy parts are not possible.
- \rightarrow Goal: sequences (g_n) such that energy of (u_n) diverges on some subset while tending to zero on another.



Energy of a potential
$$u$$
: $\int_{\Omega} \sigma |\nabla u|^2 \, dx \approx \int_{\Omega} |\nabla u|^2 \, dx \approx ||u||^2_{H^1_{\diamond}(\Omega)}$

Theoretical motivation

Calderon problem with local data:

Is σ uniquely determined by the full (local) current-to-voltage map

$$\Lambda_{\sigma}: L^2_{\diamond}(S) \to L^2_{\diamond}(S), \quad g \mapsto u|_S$$
?

(Results for measurements on ∂B : cf. Sylvester, Uhlmann, Nachmann, Astala, Päivärinta)

Monotonicity property:

$$\int_{B} (\sigma_1 - \sigma_2) |\nabla u_2|^2 \, \mathrm{d}x \ge \langle (\Lambda_{\sigma_2} - \Lambda_{\sigma_1})g, g \rangle \ge \int_{B} (\sigma_1 - \sigma_2) |\nabla u_1|^2 \, \mathrm{d}x$$

 \rightsquigarrow If $\sigma_1 - \sigma_2 > 0$ in some region where we can localize the electric energy $|\nabla u_1|^2$ then $\Lambda_{\sigma_1} \neq \Lambda_{\sigma_2}$.

"A higher conductivity in such a region can not be balanced out."





Virtual Measurements



 $f \in (H^1_\diamond(\Omega))'$: applied source on Ω $L_\Omega: (H^1_\diamond(\Omega))' \to L^2_\diamond(S), \quad f \mapsto u|_S,$ where $u \in H^1_\diamond(B)$ solves

 $\int_B \sigma \nabla u \cdot \nabla v \, \mathrm{d}x = \langle f, v |_{\Omega} \rangle \quad \text{ for all } v \in H^1_\diamond(B).$

$$\mathbf{f} \ \overline{\Omega} \subset B: \quad \nabla \cdot \sigma \nabla u = f, \quad \sigma \partial_{\nu} u |_{\partial B} = 0.$$

(UCP) yields: If $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$, $B \setminus (\overline{\Omega_1} \cup \overline{\Omega_2})$ is connected and its boundary contains *S* then $\mathcal{R}(L_{\Omega_1}) \cap \mathcal{R}(L_{\Omega_2}) = 0$.

Dual operator $L'_{\Omega}: L^2_{\diamond}(S) \to H^1_{\diamond}(\Omega), \quad g \mapsto u|_{\Omega}$, where u solves

 $\nabla \cdot \sigma \nabla u = 0, \qquad \sigma \partial_{\nu} u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else.} \end{cases}$



Some functional analysis

Lemma

Let X, Y be two reflexive Banach spaces, $A \in \mathcal{L}(X, Y)$, $y \in Y$. Then

 $y \in \mathcal{R}(A)$ iff $|\langle y', y \rangle| \le C ||A'y'|| \quad \forall y' \in Y'.$

Corollary If $||L'_{\Omega_1}g|| \leq C ||L'_{\Omega_2}g||$ for all applied currents g, i.e., $||u|_{\Omega_1}|| \leq C ||u|_{\Omega_2}||$ for the corresponding potentials u, then $\mathcal{R}(L_{\Omega_1}) \subseteq \mathcal{R}(L_{\Omega_2})$.

Contraposition If $\mathcal{R}(L_{\Omega_1}) \not\subseteq \mathcal{R}(L_{\Omega_2})$ then there exist currents (g_n) such that the corresponding potentials (u_n) satisfy

$$\|u_n|_{\Omega_1}\|_{H^1_\diamond(\Omega_1)} \to \infty \quad \text{and} \quad \|u_n|_{\Omega_2}\|_{H^1_\diamond(\Omega_2)} \to 0.$$



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Existence of localized potentials

Theorem

If $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$, $B \setminus (\overline{\Omega_1} \cup \overline{\Omega_2})$ is connected and its boundary contains S, then there exists currents (g_n) such that (the energy of) the corresponding potentials (u_n) diverges on Ω_1 while tending to zero on Ω_2 , i.e.,



Result uses only ellipticity properties, thus also holds e.g. for linear elasticity, electro- and magnetostatics.





Theoretical consequence

Corollary

Let $\sigma_1, \sigma_2 \in L^{\infty}_+(B)$ satisfy (UCP) and Λ_{σ_1} , Λ_{σ_2} be the corresponding current-to-volage-maps.

If $\sigma_2 \ge \sigma_1$ in some neighbourhood V of S and $\sigma_2 - \sigma_1 \in L^{\infty}_+(U)$ for some open $U \subseteq V$ then there exists (g_n) such that

$$\langle (\Lambda_{\sigma_2} - \Lambda_{\sigma_1}) g_n, g_n \rangle \to \infty,$$

so in particular $\Lambda_{\sigma_2} \neq \Lambda_{\sigma_1}$.



Consequence:

Piecewise constant conductivities σ are uniquely determined by Λ_{σ} (cf. e.g. Druskin, 1998, for local boundary data for a halfspace)



Construction



Construction

More constructive version of the functional analysis: Let $h \in \mathcal{R}(L_{\Omega_1})$, $h \notin \mathcal{R}(L_{\Omega_2})$. Define $\gamma_{\alpha} \in L^2_{\diamond}(S)$ by

$$\gamma_{\alpha} = (L_{\Omega_2} L_{\Omega_2}^* + \alpha I)^{-1} h, \qquad \alpha > 0.$$

Then

$$\|L'_{\Omega_2}\gamma_{\alpha}\|^2 \leq C \|L'_{\Omega_1}\gamma_{\alpha}\|$$
 and $\|L'_{\Omega_2}\gamma_{\alpha}\| \to \infty$ for $\alpha \to 0$.
So for the currents $g_{\alpha} := \frac{1}{\|L'_{\Omega_2}\gamma_{\alpha}\|^{3/2}}\gamma_{\alpha}$, the corresponding potentials u_{α} satisfy

$$\|L'_{\Omega_1}g_{\alpha}\|^2 \approx \int_{\Omega_1} \sigma |\nabla u_{\alpha}|^2 \,\mathrm{d}x \to \infty,$$
$$\|L'_{\Omega_2}g_{\alpha}\|^2 \approx \int_{\Omega_2} \sigma |\nabla u_{\alpha}|^2 \,\mathrm{d}x \to 0.$$



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Construction

Even more specific for $\sigma = 1$:

 h_z : boundary data of a electric dipole in $z \in B$, i.e., $h_z = u_z|_S$, where

$$\Delta u_z = d \cdot \nabla \delta_z, \qquad \partial_\nu u_z |_{\partial B} = 0$$

($d \in \mathbb{R}^n$, |d| = 1 fixed arbitrary direction).

- If $B \setminus \overline{\Omega}$ is connected and its boundary contains S, then for $z \notin \partial \Omega$: $h_z \in \mathcal{R}(L_{\Omega})$ iff $z \in \Omega$.
- Define currents $g_{\alpha,z}$, potentials $u_{\alpha,z}$ as on the last slide, then

$$\int_{\Omega_1} \sigma |\nabla u_{\alpha,z}|^2 \, \mathrm{d}x \to \infty, \qquad \int_{\Omega_2} \sigma |\nabla u_{\alpha,z}|^2 \, \mathrm{d}x \to 0$$



for every neighbourhood Ω_1 of z.

Numerical example



z = (0,0), color axis scaled logarithmically and cropped.

→ Electric potentials can be localized in very general domains but the problem is very ill-posed.





Special case of EIT: locate inclusions in known background medium.



Current-to-voltage map with inclusion:

 $\Lambda_1: g \mapsto u_1|_{\partial B},$

where u_1 solves $\nabla \cdot \sigma \nabla u_1 = 0$, $\partial_{\nu} u_1|_{\partial B} = g$ with $\sigma = 1 + \sigma_1 \chi_{\Omega}$, $\sigma_1 > 0$.



Current-to-voltage map without inclusion:

 $\Lambda_0: \ g \mapsto u_0|_{\partial B},$ where u_0 solves $\Delta u_0 = 0$, $\partial_{\nu} u_0|_{\partial B} = g$ with $\sigma = 1$.

Factorization method reconstructs Ω from $\Lambda := \Lambda_0 - \Lambda_1$ by using that for $z \notin \partial \Omega$:

$$h_z \in \mathcal{R}(\Lambda^{1/2})$$
 iff $z \in \Omega$.



(cf. Kirsch, Hanke, Brühl)

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Factorization Method: for $z \notin \partial \Omega$

 $z \in \Omega$ iff $h_z \in \mathcal{R}(\Lambda^{1/2})$.

Numerical implementations usually calculate regularized preimage $\psi_{z,\alpha} \approx \Lambda^{-1/2} h_z$

and use that (for $z \notin \partial \Omega$)

 $z \in \Omega$ iff $\|\psi_{z,\alpha}\|$ bounded.

 $\psi_{z,\alpha} \approx \Lambda^{-1/2} h_z$ has no physical interpretation.

Similar criterion (for $z \notin \partial \Omega$) uses $g_{z,\alpha} := \Lambda^* (\Lambda \Lambda^* + \alpha I)^{-1} h_z \approx \Lambda^{-1} h_z$

 $z \in \Omega$ iff $||L'_{\Omega}g_{z,\alpha}||$ bounded.

Physical interpretation: $L'_{\Omega}g_{z,\alpha} = u_{z,\alpha}|_{\Omega}$ with electric potential $u_{z,\alpha}$ generated by $g_{z,\alpha}$ (in homogeneous body).



Lemma

There exist
$$C, C' > 0$$
:
$$\begin{cases} |\nabla u_{z,\alpha}(z)| \leq C ||L_{\Omega}g_{z,\alpha}|| & \text{for } z \in \Omega, \\ |\nabla u_{z,\alpha}(z)| \geq C' ||L_{\Omega}g_{z,\alpha}||^2 & \text{for } z \notin \Omega. \end{cases}$$

Variant of the Factorization Method (for $z \notin \partial \Omega$):

 $z \in \Omega$ iff $|\nabla u_{z,\alpha}(z)|$ bounded.

(EIT-analogue of Arens' variant of the FM / LSM for inverse scattering).

Interpretation in context of localized potentials:

For fixed $z \notin \Omega$ (and after appropriate scaling) $u_{z,\alpha}(\cdot)$ are potentials with

energy tending to zero on Ω ,

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energy tending to infinity on every neighbourhood of z.

Numerical example

Example: $|\nabla u_{z,\alpha}(\cdot)|$ for two inclusions with conductivity jump $\sigma_1 = 1$



For fixed $z \notin \Omega$, $u_{z,\alpha}(\cdot)$ can be interpreted as a localized potential.

Plotting $|\nabla u_{z,\alpha}(\cdot)|$ may serve as a cheap alternative to the FM.



Summary and outlook

- In theory, electric potentials can be localized on quite arbitrary regions as long as they are connected to the boundary.
- In practice, such potentials can be calculated by solving ill-posed equations.
- The approximated dipole preimages of the Factorization Method (after appropriate scaling) can be interpreted as potentials that localize outside the unknown inclusions.
- Rough reconstructions can already be obtained from the preimage of a single dipole.
- Connections with the needles of Ikehata's probe method and Potthast's point source method / singular sources method?

