The Factorization Method for Real Elliptic Problems

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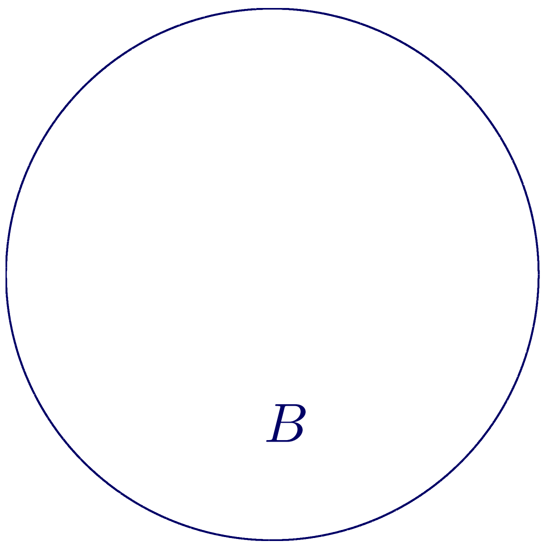
Overview

- Factorization Method for a Diffusion Example
- Extension to general real elliptic problems
- (Application: An inverse problem in Linear Elasticity)

A Diffusion Example

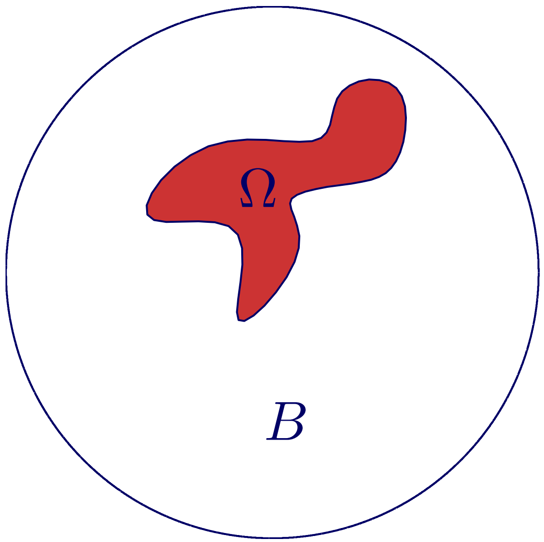


$B \subset \mathbb{R}^n$	bounded domain
$\kappa \in L^\infty_+(B)$	diffusion parameter
$c \in L^\infty_+(B)$	absorption parameter
$u \in H^1(B)$	particle density
$g \in H^{-1/2}(T)$	applied (boundary) particle flux

Boundary Measurements: $\Lambda_0 : g \mapsto u_0|_T \in H^{1/2}(T)$, where

$$\begin{aligned} -\operatorname{div}(\kappa \operatorname{grad} u_0) + c u_0 &= 0 && \text{in } B \\ \kappa \partial_\nu u_0 &= g && \text{on } T := \partial B \end{aligned}$$

Boundary Measurements



Inclusion $\Omega \subset B$ with different parameters:

$$\kappa - \kappa_1 \in L^\infty_+(\Omega)$$
$$c - c_1 \in L^\infty_+(\Omega)$$

Boundary Measurements:

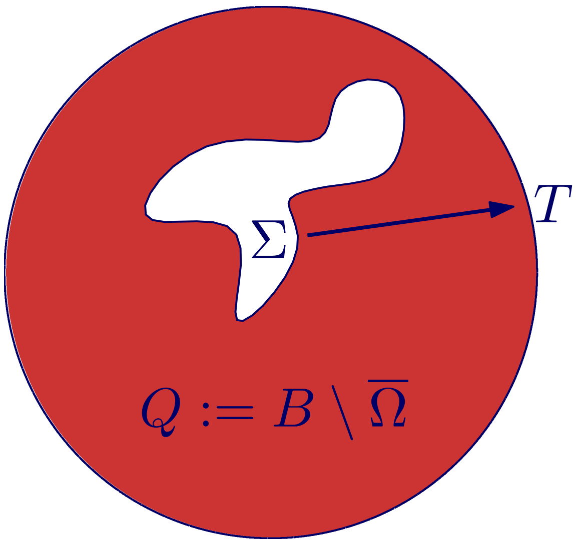
$$\Lambda_1 : H^{-1/2}(T) \to H^{1/2}(T),\ g \mapsto u_1|_T,$$

where

$$\begin{aligned} -\operatorname{div}((\kappa - \kappa_1\chi_\Omega)\operatorname{grad} u_1) + (c - c_1\chi_\Omega)u_1 &= 0 && \text{in } B \\ \kappa\partial_\nu u_1 &= g && \text{on } T := \partial B \end{aligned}$$

Goal: Reconstruct Ω from given Λ_0 and Λ_1.

Virtual Measurements



ψ: given boundary flux on $\Sigma := \partial\Omega$

$L: \ H^{-1/2}(\Sigma) \to H^{1/2}(T), \ \psi \mapsto v|_T$, where

$$\begin{aligned} -\operatorname{div}(\kappa \operatorname{grad} v) + cv &= 0 \quad \text{in } Q \quad (1) \\ \kappa \partial_\nu v &= 0 \quad \text{on } T \quad (2) \\ \kappa \partial_\nu v &= \psi \quad \text{on } \Sigma \quad (3) \end{aligned}$$

- $\mathcal{R}(L)$ determines Ω:

  $v_z|_T \in \mathcal{R}(L)$ if and only if $z \in \Omega$

  where v_z solves (1) in $B \setminus \{z\}$, v_z solves (2), v_z suff. sigular in $z \in B$.

- L, v_z independent of κ_1, c_1

Range characterization

Key identity of the Factorization Method:

$$\mathcal{R}(L) = \mathcal{R}((\Lambda_1 - \Lambda_0)^{1/2})$$

$\rightsquigarrow \mathcal{R}(L)$ (and thus Ω) can be computed from the measurements.

Such an identity

- was originally developed by Kirsch for Inverse Scattering
- is known for
  - this diffusion example (Kirsch), also with Robin B.C. (Hyvönen),
  - Electrostatics (Hähner),
  - EIT (Brühl, Hanke), also with different electrode models (Brühl, Hanke, Hyvönen) and in the half space (Schappel).

Can it be extended to general real elliptic problems?

Framework for Diffusion Example

$\Lambda_0: \; g \mapsto u_0|_T$, where $\left\{ \begin{array}{rcll} -\operatorname{div}(\kappa \operatorname{grad} u_0) + cu_0 & = & 0 & \text{in } B \\ \kappa \partial_\nu u_0 & = & g & \text{on } T := \partial B \end{array} \right.$

- $\boxed{\Lambda_0 = \gamma_{B\to T} A_0^{-1} \gamma'_{B\to T},}$ where

$$\begin{aligned} \gamma_{B\to T}: &\quad H^1(B) \to H^{1/2}(T), \quad u \mapsto u|_T, \\ A_0: &\quad H^1(B) \to H^1(B)', \quad \langle A_0 u, v\rangle = \int_B (\kappa \nabla u \nabla v + cuv)\, \mathrm{d}x \end{aligned}$$

Analogously

- $\boxed{\Lambda_1 = \gamma_{B\to T} A_1^{-1} \gamma'_{B\to T},}$ where $A_1: \; H^1(B) \to H^1(B)',$

$$\langle A_1 u, v\rangle = \int_B ((\kappa - \kappa_1 \chi_\Omega) \nabla u \nabla v + (c - c_1 \chi_\Omega) uv)\, \mathrm{d}x$$

Framework for Diffusion Example

A_0, A_1 share a common part:

$$A_0 = E'_Q A_Q E_Q + E'_\Omega A_{\Omega,0} E_\Omega$$
$$A_1 = E'_Q A_Q E_Q + E'_\Omega A_{\Omega,1} E_\Omega,$$

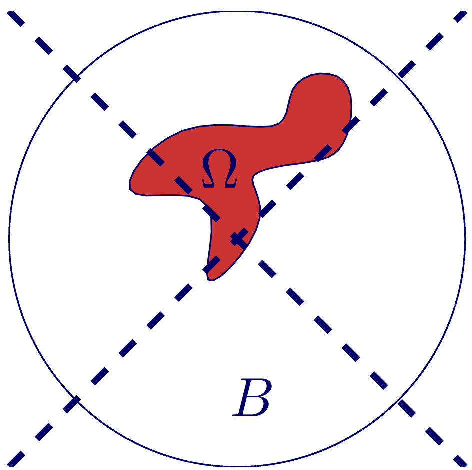
where $E_Q : H^1(B) \to H^1(Q), u \mapsto u|_Q$ (E_Ω analogously) and

$$\langle A_Q u, v\rangle = \int_Q (\kappa \nabla u \nabla v + cuv)\, \mathrm{d}x$$
$$\langle A_{\Omega,0} u, v\rangle = \int_\Omega (\kappa \nabla u \nabla v + cuv)\, \mathrm{d}x$$
$$\langle A_{\Omega,1} u, v\rangle = \int_\Omega ((\kappa - \kappa_1) \nabla u \nabla v + (c - c_1)uv)\, \mathrm{d}x$$

Virtual measurements:

- $$L = \gamma_{Q\to T} A_Q^{-1} \gamma'_{Q\to\Sigma},$$

A general setting



Given real Hilbert spaces

$$H(B),\ H(Q),\ H(\Omega),\ H(T),\ H(\Sigma)$$

with suitable restriction operators

$$E_{\mathcal{X}}:\ H(B) \to H(\mathcal{X}),\ \mathcal{X} \in \{Q, \Omega\}$$

and suitable trace operators

$$\gamma_{\mathcal{X}\to\mathcal{Y}}:\ H(X) \to H(Y),$$
$$(\mathcal{X}, \mathcal{Y}) \in \{(B,T), (Q,T), (Q,\Sigma), (\Omega,\Sigma)\},$$

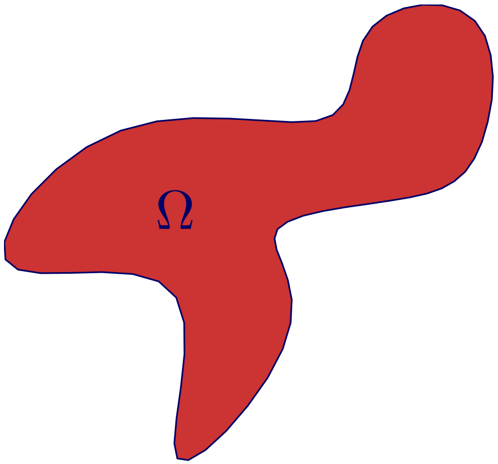
let $A_Q:\ H(Q) \to H(Q)'$, $A_{\Omega,0}, A_{\Omega,1}:\ H(\Omega) \to H(\Omega)'$ be continuous, symmetric and coercive and define ($i = 0, 1$)

$$A_i := E_Q' A_Q E_Q + E_\Omega' A_{\Omega,i} E_\Omega, \qquad \Lambda_i := \gamma_{B\to T} A_i^{-1} \gamma_{B\to T}',$$

and $L := \gamma_{Q\to T} A_Q^{-1} \gamma_{Q\to\Sigma}'$

When does $\mathcal{R}(L) = \mathcal{R}((\Lambda_1 - \Lambda_0)^{1/2})$ hold?

Main Result



Theorem

If $A_{\Omega,0} - A_{\Omega,1}$ is coercive, then

$$\mathcal{R}((\Lambda_1 - \Lambda_0)^{1/2}) = \mathcal{R}(L).$$

- To take the square root, $H(T)$ is identified with its dual.
- Diffusion Example:

$$\langle (A_{\Omega,0} - A_{\Omega,1})u, v\rangle = \int_\Omega (\kappa_1 \nabla u \nabla v + c_1 uv)\,\mathrm{d}x$$

$\rightsquigarrow A_{\Omega,0} - A_{\Omega,1}$ coercive if $\kappa_1, c_1 \in L^\infty_+(\Omega)$.

Sketch of the Proof

Theorem

If $A_{\Omega,0} - A_{\Omega,1}$ *is coercive, then*

$$\mathcal{R}((\Lambda_1 - \Lambda_0)^{1/2}) = \mathcal{R}(L).$$

- Factorization: $\Lambda_1 - \Lambda_0 = LFL'$, where

$$F = (\gamma^{-}_{Q\to\Sigma})' A_Q E_Q (A_1^{-1} - A_0^{-1}) E'_Q A_Q \gamma^{-}_{Q\to\Sigma}.$$

  $\gamma^{-}_{Q\to\Sigma}$: continuous right inverse of $\gamma_{Q\to\Sigma}$

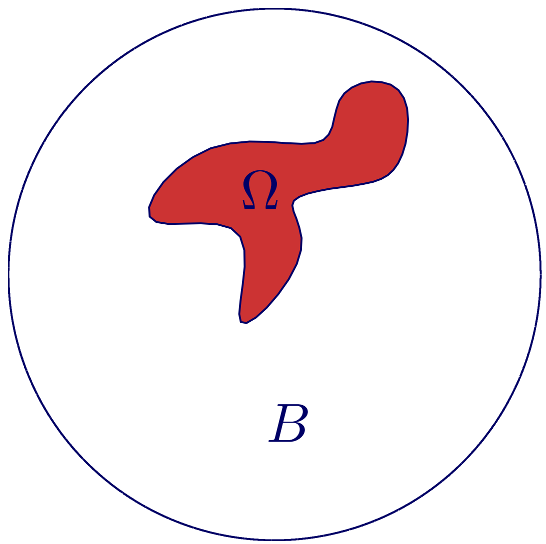
- $(A_1^{-1} - A_0^{-1}) \geq A_0^{-1} E'_\Omega (A_{\Omega,0} - A_{\Omega,1}) E_\Omega A_0^{-1}$

  $\rightsquigarrow F$ coercive if $A_{\Omega,0} - A_{\Omega,1}$ coercive.

- As $A^*A = B^*B$ implies $\mathcal{R}(A^*) = \mathcal{R}(B^*)$:

$$\mathcal{R}((\Lambda_1 - \Lambda_0)^{1/2}) = \mathcal{R}(LF^{1/2}) = \mathcal{R}(L).$$

Linear Elasticity



λ, μ: Lamé-constants

u: displacement

g: applied (boundary) forces

$e(u) := \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)_{ij}$ strain tensor

$I := (\delta_{ij})_{ij}$

Boundary Measurements: $\Lambda_i : \; g \longrightarrow u_i|_T$, where

$$\begin{aligned} \operatorname{div}((\lambda - \lambda_1 \chi_\Omega)(\operatorname{tr} e(u))I + 2(\mu - \mu_1 \chi_\Omega)e(u)) &= 0 \quad \text{in } B \\ (\lambda(\operatorname{tr} e(u))I + 2\mu e(u))\nu &= g \quad \text{on } T = \partial B \end{aligned}$$

$i = 1:$ with the terms $\lambda_1 \chi_\Omega, \mu_1 \chi_\Omega$

$i = 0:$ without

Linear Elasticity

Fits into our generalized framework with

$$\begin{aligned}\langle A_Q u, v\rangle &= \int_Q (\lambda \operatorname{tr} e(u) \operatorname{tr} e(v) + 2\mu e(u) : e(v))\, \mathrm{d}x\\ \langle A_{\Omega,0} u, v\rangle &= \int_\Omega (\lambda \operatorname{tr} e(u) \operatorname{tr} e(v) + 2\mu e(u) : e(v))\, \mathrm{d}x\\ \langle A_{\Omega,1} u, v\rangle &= \int_\Omega ((\lambda - \lambda_1) \operatorname{tr} e(u) \operatorname{tr} e(v) + 2(\mu - \mu_1) e(u) : e(v))\, \mathrm{d}x\end{aligned}$$

($H(B)$, $H(T)$, $\ldots$ *suitable factor spaces of* $H^1(B)^3$, $H^{1/2}(T)^3$, $\ldots$)

$$\langle (A_{\Omega,0} - A_{\Omega,1}) u, v\rangle = \int_\Omega (\lambda_1 \operatorname{tr} e(u) \operatorname{tr} e(v) + 2\mu_1 e(u) : e(v))\, \mathrm{d}x$$

$\leadsto$ If $\lambda_1 \geq 0$, $\mu_1 \in L^\infty_+(\Omega)$ then $A_{\Omega,0} - A_{\Omega,1}$ is coercive and thus

$$\boxed{\mathcal{R}((\Lambda_1 - \Lambda_0)^{1/2}) = \mathcal{R}(L).}$$

($\leadsto z \in \Omega$ *iff* $v_z \in \mathcal{R}(L) = \mathcal{R}((\Lambda_1 - \Lambda_0)^{1/2})$ *with suitable test functions* v_z.)

Summary

- Factorization Method can be generalized to real elliptic problems
- Range characterization holds if the inclusions satisfy a coercivity condition.
- Coercivity condition can easily be translated into conditions on the coefficients of a given elliptic equation.