Monotony Based Imaging in EIT

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Abstract. We consider the problem of determining conductivity anomalies inside a body from voltage-current measurements on its surface. By combining the monotonicity method of Tamburrino and Rubinacci with the concept of localized potentials, we derive a new imaging method that is capable of reconstructing the exact (outer) shape of the anomalies. We furthermore show that the method can be implemented without solving any non-homogeneous forward problems and show a first numerical result.

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INTRODUCTION

Electrical impedance tomography (EIT) aims to image the conductivity σ within an electrically conducting subject (e.g., the human body) from current-voltage measurements on the subject's boundary. Mathematically, this can be stated as the problem of reconstructing the coefficient $\sigma(x)$ in the elliptic partial differential equation

$$\nabla \cdot \sigma(x) \nabla u(x) = 0, \quad x \in \Omega, \tag{1}$$

from knowledge of the Neumann-to-Dirichlet operator

$$\Lambda(\sigma) : \; g \mapsto u|_{\partial\Omega},$$

where u solves (1) with Neumann boundary values $\sigma \partial_\nu u|_{\partial\Omega} = g$. For fixed $\sigma \in L^\infty_+(\Omega)$, this is a compact and self-adjoint linear operator from $L^2_\diamond(\partial\Omega)$ to $L^2_\diamond(\partial\Omega)$, where the subscript "$+$", resp., "$\diamond$", denotes the subspace of functions with positive (essential) infima, resp., vanishing integral mean on $\partial\Omega$.

Herein, we concentrate on the following anomaly detection (or shape detection) problem in EIT. Assume that σ differs from a known reference conductivity σ_0 only in a (possibly disconnected) open subset D with $\overline{D} \subset \Omega$,

$$\sigma(x) = \sigma_0 + \sigma_D(x)\chi_D(x) \in L^\infty_+(\Omega),$$

where $\chi_D(x)$ denotes the characteristic function of D. Our goal is to locate the conductivity anomalies D from $\Lambda(\sigma)$.

For ease of presentation, we thereby assume that the reference conductivity is constant, $\sigma_0 > 0$ and that there is a clear jump between the anomaly and the background, i.e., $\sigma_D \in L^\infty_+(D)$. We furthermore assume that $\Omega \setminus \overline{D}$ is connected. Our results can be extended to known inhomogeneous reference conductivities as long as a unique continuation principle is satisfied. Also, we could allow smooth transitions between the background and the anomalies, i.e., $\sigma_D \geq 0$, by considering interior points as in [1] and allow for holes in D by introducing the concept of the outer support as in [2, 3].

In this work we show that D can be reconstructed by a simple monotony test. Our result extends the monotonicity method of Tamburrino and Rubinacci [4, 5] who obtained upper and lower bounds for the anomalies. Our main tool is the method of localized potentials developed by one of the authors in [6]. Furthermore, we also show how to implement the monotonicity test in a very efficient way, i.e., without solving any non-homogeneous forward problems.

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MONOTONY BASED IMAGING

A monotony relation

There exists a monotony relation between the conductivity and the Neumann-Dirichlet-operator. For two conductivities, $\sigma_1, \sigma_2 \in L^\infty_+(\Omega)$

$$\int_\Omega (\sigma_2 - \sigma_1)|\nabla u_2|^2 \,\mathrm{d}x \le \int_{\partial\Omega} g\,(\Lambda(\sigma_1) - \Lambda(\sigma_2))\,g\,\mathrm{d}s \le \int_\Omega \frac{\sigma_2}{\sigma_1}(\sigma_2 - \sigma_1)|\nabla u_2|^2 \,\mathrm{d}x, \tag{2}$$

where u_2 solves (1) with conductivity σ_2 and Neumann boundary values $\sigma_2 \partial_\nu u_2|_{\partial\Omega} = g$, cf., e.g., [3, Lemma 2.1] for a proof of (2).

A simple consequence is that

$$\sigma_1 \le \sigma_2 \quad \Longrightarrow \quad \Lambda(\sigma_1) \ge \Lambda(\sigma_2). \tag{3}$$

The first inequality is to be understood in a pointwise sense, i.e., $\sigma_1(x) \ge \sigma_2(x)$ for $x \in \Omega$ a.e. The second inequality then holds in the sense of quadratic forms on $L^2_\diamond(\partial\Omega)$, i.e. $\int_{\partial\Omega} g\Lambda(\sigma_1)g\,\mathrm{d}s \le \int_{\partial\Omega} g\Lambda(\sigma_2)g\,\mathrm{d}s$.

Physically speaking, the integral $\int_{\partial\Omega} g\Lambda(\sigma_1)g\,\mathrm{d}s$ is the power that is needed to maintain the boundary current g in a domain with conductivity σ_1, or, equivalently, the electric power absorbed in the domain when applying the current g. Hence, (2) corresponds to the simple fact, that a domain with a smaller conductivity absorbs more electric power than one with a larger conductivity, no matter what current pattern is applied to the boundary.

Monotony probing

The monotony relation (3) motivates to probe the domain with small test anomalies. Let $B \subset \Omega$ be a subdomain of Ω, e.g., a small ball, and let $\chi_B(x)$ be its characteristic function. Choose a constant σ_B such that $0 < \sigma_B \le \inf_{x \in D} \sigma_D(x)$ (for a faster implementation we later require a smaller σ_B, see (5) below), and define a test conductivity $\kappa(x) := \sigma_0 + \sigma_B \chi_B(x)$. Then, by monotony, we have that

$$B \subseteq D \quad \Longrightarrow \quad \kappa \le \sigma \quad \Longrightarrow \quad \Lambda(\kappa) \ge \Lambda(\sigma). \tag{4}$$

Hence, we could probe or sample the domain Ω for the unknown anomalies by calculating the Neumann-Dirichlet operators $\Lambda(\kappa)$ for a large number of test domains B and mark the area of all the test domains for which $\Lambda(\kappa) \ge \Lambda(\sigma)$. This would give an upper bound on the unknown anomaly D. This approach goes back to Tamburrino and Rubinacci [4, 5] who combine it with the idea of probing also with larger test domains which then gives a lower bound for D.

A converse monotony relation

Using the concept of localized potentials, we can show that a converse of (4) is true. If $B \not\subseteq D$ then there is some overlapping part $B \setminus \overline{D}$, that can be reached from the boundary, cf. figure 1. [6, Thm. 2.7] then assures that there exists a current g such that the corresponding absorbed power is very large in this overlapping part, but very small in the rest of B and D (the so-called *localized potential*). Hence, when this special current is applied, less power is absorbed in a domain with test conductivity κ then in a domain with conductivity σ, so that $\Lambda(\kappa) \not\ge \Lambda(\sigma)$. Consequently, we obtain that

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(\kappa) \ge \Lambda(\sigma).$$

This shows that probing the domain with small test domains already gives (in the limit of vanishing diameter of the test domains) the exact shape D.

Fast probing by linearization

For each test domain B, we have a different $\kappa(x) = \sigma_0 + \sigma_B \chi_B(x)$, so that the monotony test using $\Lambda(\kappa)$ would require to solve the elliptic partial differential equation (1) with a new conductivity κ. A much faster method is obtained by

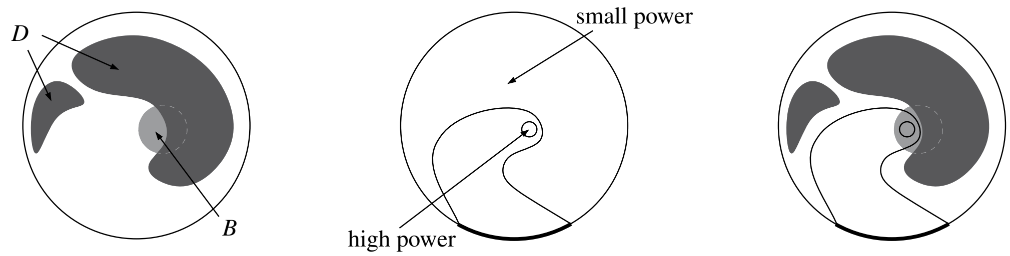

FIGURE 1. If some part of the test domain B does not belong to the true anomaly D (left picture) then there exists a localized potential with high power in this overlapping part and low power in the rest of B and D (middle and right picture).

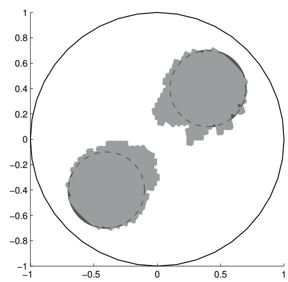

FIGURE 2. Reconstruction of two circular inclusions with the proposed method.)

replacing $\Lambda(\kappa)$ with its linear approximation $\Lambda(\kappa) \approx \Lambda(\sigma_0) + \Lambda'(\sigma_0)(\kappa - \sigma_0)$. $\Lambda'(\sigma_0)$ denotes the Fréchet-derivative of Λ evaluated at σ_0. Its associated quadratic form can be written as

$$\int_{\partial\Omega} g\Lambda'(\sigma_0)(\kappa - \sigma_0)g \, \mathrm{d}s = -\int_B \sigma_B |\nabla u_0|^2 \, \mathrm{d}x$$

where u_0 solves (1) with conductivity σ_0 and Neumann boundary values $\sigma_0 \partial_\nu u_0|_{\partial\Omega} = g$. Note that $\Lambda'(\sigma_0)(\kappa - \sigma_0)$ can be efficiently computed for different κ as it only requires the solution of (1) with homogeneous conductivity σ_0.

If we choose σ_B so small that

$$0 < \sigma_B \leq \sigma_0 \frac{\inf_{x\in D} \sigma_D(x)}{\sup_{x\in\Omega} \sigma(x)}, \tag{5}$$

then (2) still yields

$$B \subseteq D \quad \Longrightarrow \quad \Lambda(\sigma_0) + \Lambda'(\sigma_0)(\kappa - \sigma_0) \geq \Lambda(\sigma_1).$$

Using the technique of localized potentials as above we obtain that also the converse holds true, so that

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(\sigma_0) + \Lambda'(\sigma_0)(\kappa - \sigma_0) \geq \Lambda(\sigma_1). \tag{6}$$

Hence, we can reconstruct D by probing the domain Ω with small test domains B and marking the area of all test domains for which the right side of (6) holds true. In the limit of vanishing diameter of the test domains this gives the exact position and shape of the anomaly D.

NUMERICAL RESULTS

Figure 2 shows a first numerical result that we obtained with the method described above. The dark grey color and dashed lines show the position of two circular inclusions. The upper right inclusion has a constant conducitivity contrast of 1. In the lower left inclusion the contrast varies (linearly with radius) from 2 in the center to 0.45 on the boundary. The background conductivity is set to $\sigma_0 = 1$. For the test domains we used $\sigma_B = 0.15$, which is the largest number that fulfills (5). The test domains for which the right hand side of (6) holds true are filled with light grey color.

REFERENCES

1. B. Gebauer, and N. Hyvönen: Factorization method and irregular inclusions in electrical impedance tomography, *Inverse Problems* **23**, 2159–2170 (2007).
2. B. Gebauer, and N. Hyvönen: Factorization method and inclusions of mixed type in an inverse elliptic boundary value problem, *Inverse Probl. Imaging* **2**, 355–372 (2008).
3. B. Harrach, and J. K. Seo: Exact shape-reconstruction by one-step linearization in electrical impedance tomography, *SIAM J. Math. Anal.* (to appear).
4. A. Tamburrino, and G. Rubinacci: A new non-iterative inversion method for electrical resistance tomography, *Inverse Problems* **18**, 1809–1829 (2002).
5. A. Tamburrino: Monotonicity based imaging methods for elliptic and parabolic inverse problems, *J. Inverse Ill-Posed Probl.* **14**, 633–642 (2006).
6. B. Gebauer: Localized potentials in electrical impedance tomography, *Inverse Probl. Imaging* **2**, 251–269 (2008).