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ON LOCALIZING AND CONCENTRATING ELECTROMAGNETIC FIELDS*

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Abstract. We consider field localizing and concentration of electromagnetic waves governed by the time-harmonic anisotropic Maxwell system in a bounded domain. It is shown that there always exist certain boundary inputs which can generate electromagnetic fields with the energy localized/concentrated in a given subdomain while nearly vanishing in another given subdomain. The theoretical results may have potential applications in telecommunication, inductive charging, and medical therapy. We also derive a related Runge approximation result for the time-harmonic anisotropic Maxwell system with partial boundary data.

Key words. electromagnetic waves, localizing and concentration, anisotropic Maxwell system, Runge approximation, partial data

AMS subject classifications. Primary, 35Q61; Secondary, 78A25, 78A45

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1. Introduction.

1.1. Background and motivation.

The electromagnetic (EM) phenomena are ubiquitous and they lie at the heart of many scientific and technological applications including radar and sonar, geophysical exploration, medical imaging, information processing, and communication. In this paper, we are mainly concerned with the mathematical study of field localizing and concentration of electromagnetic waves governed by the time-harmonic Maxwell system in a bounded anisotropic medium. More specifically, we show that there always exist certain boundary inputs which can generate the desired electromagnetic fields that are localized/concentrated in a given subdomain while nearly vanishing in another given subdomain.

The localizing and concentration of electromagnetic fields can have many potential applications. In telecommunication [44], one common means of transmitting information between communication participants is via the electromagnetic radiation. In a certain practical scenario, say, secure communication, one may intend the information to be transmitted mainly to a partner located at a certain region, while avoiding the transmission to another region. Clearly, if the information is encoded into the electromagnetic waves that are localized and concentrated in the region where the partner is located while nearly vanishing in the undesired region, then one can achieve the expected telecommunication effect. In the setup of our proposed study, one can easily obtain the aforementioned communication effect, in particular if the communication participants transmit and receive information on some surface patches.

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Concentrating electromagnetic fields can also be useful in inductive charging, also known as wireless charging or cordless charging [43], which is an emerging technology that can have significant impact on real life. It uses electromagnetic fields to transfer energy between two objects through electromagnetic induction. Clearly, the energy transfer would be more efficient and effective if the corresponding electromagnetic fields are concentrated around the charging station. The localizing of electromagnetic fields can also have potential application in electromagnetic therapy. Though it is mainly considered to be pseudoscientific with no affirmative evidence, electromagnetic therapy has been widely practiced and claims to treat disease by applying electromagnetic radiation to the body. If electromagnetic therapy shall be proven to be effective, then through the use of certain purposely designed sources, one can generate electromagnetic fields that are concentrated around the diseased area.

The above conceptual and potential applications make the study of field concentration and localizing very appealing. Nevertheless, it is emphasized that in the current article, we are mainly concerned with mathematical and theoretical study. We achieve some substantial progress on this interesting topic, though the corresponding study is by no means complete. It is also interesting to note that the localizing of resonant electromagnetic fields has been used to produce invisibility cloaking and has received significant attention in the literature in recent years [2, 3, 6, 31, 34, 37]. The corresponding study is mainly based on the use of plasmonic materials to induce so-called anomalous localized resonance.

Our mathematical argument for proving the existence of localized and concentrated electromagnetic fields is mainly based on combining the unique continuation property (UCP) for the anisotropic Maxwell system with a functional analytic duality argument developed in [12]. By a similar argument, we also obtain a related Runge approximation property.

The use of blow up solutions has a long tradition in the study of inverse boundary value problems; cf. [1, 25, 27, 28] for early seminal works on this topic. Moreover, the combination of localized fields and monotonicity relations has led to the development of monotonicity-based methods for obstacle/inclusion detection; cf. [22, 39] for the origins and mathematical justification of this approach, [5, 7, 9, 10, 11, 16, 17, 18, 19, 21, 23, 33, 38, 40, 42, 45] for further recent contributions, and the recent works [13, 20] for the Helmholtz equation. Theoretical uniqueness results for inverse coefficient problems have also been obtained by this approach in [4, 14, 15, 21, 24].

In this work, we show the existence of localized electromagnetic fields for the more challenging case of a time-harmonic anisotropic Maxwell system with partial data. We also derive a Runge approximation result, which shows that every solution in a subdomain can be approximated by a solution on the whole domain. In that context let us note the famous equivalence theorem from Peter Lax [29]: the weak UCP is equivalent to the Runge approximation property for the second order elliptic equation. In our study, we affirmatively verify that this property still holds for the anisotropic Maxwell system.

The rest of this section is devoted to the mathematical description of the setup of our study and the statement of the main result.

1.2. Mathematical setup and statement of the main result. Let Ω be a simply connected domain in $\mathbb{R}^3$ with a Lipschitz connected boundary $\partial\Omega$. Let $\epsilon = (\epsilon_{ij})_{1\le i,j\le 3}$ and $\mu = (\mu_{ij})_{1\le i,j\le 3}$ be two 3×3 real matrix-valued functions on Ω satisfying the following:

- Strong ellipticity: There exist constants $\mu_0 > 0$ and $\epsilon_0 > 0$ verifying

$$
\begin{cases} \mu_0|\xi|^2 \le \sum_{i,j=1}^3 \mu_{ij}(x)\xi_i\xi_j \le \mu_0^{-1}|\xi|^2, \\ \epsilon_0|\xi|^2 \le \sum_{i,j=1}^3 \epsilon_{ij}(x)\xi_i\xi_j \le \epsilon_0^{-1}|\xi|^2 \end{cases} \quad \text{for any } x \in \Omega \text{ and } \xi \in \mathbb{R}^3. \tag{1.1}
$$

- Smoothness: ϵ and μ are Lipschitz continuous.
- Symmetry: ϵ and μ are symmetric matrices, that is, $\epsilon_{ij} = \epsilon_{ji}$ and $\mu_{ij} = \mu_{ji}$ for all $i, j = 1, 2, 3$.

The functions ϵ and μ, respectively, signify the electric permittivity and magnetic permeability of the medium in Ω. Consider the time-harmonic electromagnetic wave propagation in Ω. With the e^{-ikt} time-harmonic convention assumed, we let $E(x)$ and $H(x)$, respectively, denote the electric and magnetic fields. Here, $k \in \mathbb{R}_+$ signifies a circular frequency. Then the electromagnetic wave propagation is governed by the following Maxwell system:

$$
\begin{cases} \nabla \times E - ik\mu H = 0 & \text{in } \Omega, \\ \nabla \times H + ik\epsilon E = 0 & \text{in } \Omega, \\ \nu \times E = \begin{cases} f & \text{on } \Gamma, \\ 0 & \text{otherwise} \end{cases} & \text{on } \partial\Omega, \end{cases} \tag{1.2}
$$

where Γ is an arbitrary nonempty relatively open subset of $\partial\Omega$ and ν is the unit outer normal vector on $\partial\Omega$. It is assumed that $k > 0$ is not an eigenvalue (or nonresonance, see section 2) for (1.2) and $f \in C_c^\infty(\Gamma)$ throughout this paper.

The main result concerning the localized electromagnetic fields for the anisotropic Maxwell system (1.2) is contained in the following theorem.

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $\Gamma \subseteq \partial\Omega$ be a relatively open subset of the boundary. Let $\epsilon, \mu \in L^\infty(\Omega, \mathbb{R}^{3\times 3})$ be real-valued, piecewise Lipschitz continuous functions satisfying* (1.1) *and $k > 0$ be a nonresonant frequency. Let $D \Subset \Omega$ be a closed set with a connected complement $\Omega \setminus D$. For every open set $M \subseteq \Omega$ with $M \not\subseteq D$ (see Figure 1 for the schematic illustration), there exists a sequence $\{f^{(\ell)}\}_{\ell\in\mathbb{N}} \subset C_c^\infty(\Gamma)$ such that the electromagnetic fields fulfill*

$$
\int_M \left(|E^{(\ell)}|^2 + |H^{(\ell)}|^2\right) \mathrm{d}x \to \infty \quad \textit{and} \quad \int_D \left(|E^{(\ell)}|^2 + |H^{(\ell)}|^2\right) \mathrm{d}x \to 0 \textit{ as } \ell \to \infty,
$$

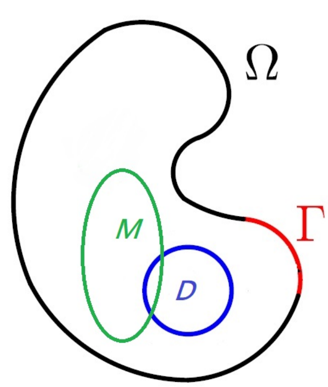

FIG. 1. *Schematic illustration of the field localizing and concentration.*

where, for $\ell \in \mathbb{N}$, $(E^{(\ell)}, H^{(\ell)}) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$ *is a solution of*

$$\begin{cases} \nabla \times E^{(\ell)} - ik\mu H^{(\ell)} = 0 & \text{in } \Omega, \\ \nabla \times H^{(\ell)} + ik\epsilon E^{(\ell)} = 0 & \text{in } \Omega \end{cases}$$

with the boundary data

$$\nu \times E^{(\ell)}|_{\partial\Omega} = \begin{cases} f^{(\ell)} & \text{on } \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1.1. We denote the sequence $\{(E^{(\ell)}, H^{(\ell)})\}_{\ell \in \mathbb{N}}$ in Theorem 1.1 to be the localized electromagnetic fields.

The rest of the paper is organized as follows. In section 2, we present results on the well-posedness of the time-harmonic anisotropic Maxwell system. We also provide the UCP for the anisotropic Maxwell system, whenever the coefficients μ and ϵ are piecewise Lipschitz continuous matrix-valued functions. In section 3, we demonstrate that there exist localized electromagnetic fields, which proves Theorem 1.1. The method relies on certain functional analysis techniques. In section 4 we prove a related Runge approximation property for the anisotropic Maxwell system with partial boundary data.

2. The anisotropic Maxwell system in a bounded domain. In this section, we summarize some useful results of the Maxwell system, including the unique solvability and a UCP. Throughout this section we let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain.

2.1. Spaces and traces. We introduce the spaces

$$H(\mathrm{div}, \Omega) := \left\{ E \in L^2(\Omega)^3; \quad \nabla \cdot E \in L^2(\Omega) \right\},$$
$$H(\mathrm{curl}, \Omega) := \left\{ E \in L^2(\Omega)^3; \quad \nabla \times E \in L^2(\Omega)^3 \right\},$$

and the tangential trace operators

$$\gamma_t : H(\mathrm{curl}, \Omega) \to H^{-1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega), \quad E \mapsto \gamma_t E := \nu \times E|_{\partial\Omega},$$
$$\gamma_T : H(\mathrm{curl}, \Omega) \to H^{-1/2}(\mathrm{curl}_{\partial\Omega}, \partial\Omega), \quad E \mapsto \gamma_T E := \nu \times (E|_{\partial\Omega} \times \nu),$$

where, and also in what follows, all functions are complex-valued unless indicated otherwise. γ_t and γ_T are surjective bounded linear operators with bounded right inverses γ_t^{-1} and and γ_T^{-1} (cf. [8, 41]). The space $H^{-1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega)$ can be identified with the dual of $H^{-1/2}(\mathrm{curl}_{\partial\Omega}, \partial\Omega)$, and for all $E, F \in H(\mathrm{curl}, \Omega)$ we have the integration by parts formula

$$\int_\Omega (\nabla \times E) \cdot F \,\mathrm{d}x - \int_\Omega E \cdot (\nabla \times F) \,\mathrm{d}x = \int_{\partial\Omega} (\nu \times E|_{\partial\Omega}) \cdot (\nu \times (F|_{\partial\Omega} \times \nu)) \,\mathrm{d}S \tag{2.1}$$

(cf. [8, 35]), where the dual pairing on $H^{-1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega) \times H^{-1/2}(\mathrm{curl}_{\partial\Omega}, \partial\Omega)$ is written as an integral for notational convenience.

The subspace of $H(\mathrm{curl}, \Omega)$-functions with vanishing tangential traces is denoted by

$$H_0(\mathrm{curl}, \Omega) := \{E \in H(\mathrm{curl}, \Omega) : \ \nu \times E|_{\partial\Omega} = 0\}.$$

$H_0(\text{curl},\Omega)$ is a closed subspace of $H(\text{curl},\Omega)$ and $C_0^\infty(\Omega)^3$ is dense in $H_0(\text{curl},\Omega)$ (cf. [35]).

To treat partial boundary data on a relatively open subset $\Gamma \subseteq \partial\Omega$, we also introduce the space of functions on Γ that can be extended by zero to the trace of a $H(\text{curl},\Omega)$-function

$$H(\Gamma) := \text{closure of } C_c^\infty(\Gamma) \text{ in } H^{-1/2}(\text{div}_{\partial\Omega},\partial\Omega). \tag{2.2}$$

For all $E \in H(\text{curl},\Omega)$, we identify the restricted trace $\nu\times(E|_\Gamma\times\nu)$ with the quotient space element

$$\nu\times(E\times\nu)|_{\partial\Omega} + H(\Gamma)^\perp \in H^{-1/2}(\text{curl}_{\partial\Omega},\partial\Omega)/H(\Gamma)^\perp = H(\Gamma)^*,$$

and thus define the restricted trace operator

$$\gamma_T^{(\Gamma)}: \; H(\text{curl},\Omega)\to H(\Gamma)^*, \quad E\mapsto \gamma_T^{(\Gamma)}E := \nu\times(E|_\Gamma\times\nu).$$

2.2. Well-posedness of the anisotropic Maxwell system. Given anisotropic coefficients $\epsilon,\mu \in L^\infty(\Omega,\mathbb{R}^{3\times3})$ satisfying (1.1), $k>0$, $J,K\in L^2(\Omega)^3$, and $f\in H^{-1/2}(\text{div}_{\partial\Omega},\partial\Omega)$, we consider the Maxwell system for $(E,H)\in H(\text{curl},\Omega)\times H(\text{curl},\Omega)$ such that

$$\nabla\times E - ik\mu H = K \quad \text{in } \Omega, \tag{2.3}$$
$$\nabla\times H + ik\epsilon E = J \quad \text{in } \Omega, \tag{2.4}$$
$$\nu\times E|_{\partial\Omega} = f. \tag{2.5}$$

For the variational formulation of (2.3)–(2.4) we introduce the sesquilinear form

$$\mathcal{B}: \; H(\text{curl},\Omega)\times H(\text{curl},\Omega)\to\mathbb{C},$$
$$\mathcal{B}(E,F) := \int_\Omega \left(\mu^{-1}\nabla\times E\right)\cdot\left(\nabla\times\overline{F}\right)\,\mathrm{d}x - \int_\Omega k^2\epsilon E\cdot\overline{F}\,\mathrm{d}x.$$

Then we have the following variational formulation and well-posedness result.

THEOREM 2.1.

(a) $(E,H)\in H(\text{curl},\Omega)\times H(\text{curl},\Omega)$ *solve* (2.3)–(2.4) *if and only if* $E\in H(\text{curl},\Omega)$ *solves*

$$\mathcal{B}(E,F) = \int_\Omega ikJ\cdot\overline{F}\,\mathrm{d}x + \int_\Omega \left(\mu^{-1}K\right)\cdot\left(\nabla\times\overline{F}\right)\,\mathrm{d}x \quad \textit{for all } F\in H_0(\text{curl},\Omega),$$

and $H = -\frac{i}{k}\mu^{-1}(\nabla\times E - K)$.

(b) *The set of* $k>0$ *for which the homogeneous system* (2.3)–(2.5) *with* $J=0$, $K=0$, *and* $f=0$ *possesses a nontrivial solution is discrete. We call these* k resonance frequencies.

(c) *If* k *is not a resonance frequency, then there exists a unique solution* $(E,H)\in H(\text{curl};\Omega)\times H(\text{curl};\Omega)$ *of* (2.3)–(2.5), *and the solution depends linearly and continuously on* $J,K\in L^2(\Omega)^3$ *and* $f\in H^{-1/2}(\text{div}_{\partial\Omega},\partial\Omega)$.

The proof of Theorem 2.1 follows from a standard argument. Since we couldn't find a convenient reference for precisely this setting, we supply a proof for the sake of completeness. We first derive the following lemma.

LEMMA 2.1. $(E,H) \in H(\mathrm{curl},\Omega) \times H(\mathrm{curl},\Omega)$ *solves* (2.3)–(2.4) *if and only if* $E \in H(\mathrm{curl},\Omega)$ *solves*

$$\mathcal{B}(E,F) = \int_\Omega ikJ \cdot \overline{F}\,\mathrm{d}x + \int_\Omega \left(\mu^{-1}K\right) \cdot \left(\nabla \times \overline{F}\right)\,\mathrm{d}x \quad \textit{for all } F \in H_0(\mathrm{curl},\Omega), \tag{2.6}$$

and $H = -\frac{i}{k}\mu^{-1}(\nabla \times E - K)$.

Proof. Let $(E,H) \in H(\mathrm{curl},\Omega) \times H(\mathrm{curl},\Omega)$ solve (2.3)–(2.4). Then (2.3) implies that

$$E \in H(\mathrm{curl},\Omega) \quad \text{and} \quad H = -\frac{i}{k}\mu^{-1}(\nabla \times E - K),$$

and combining (2.3) and (2.4) we obtain

$$\nabla \times \left(\mu^{-1}\left(\nabla \times E - K\right)\right) - k^2\epsilon E = ikJ, \tag{2.7}$$

which also shows that $\mu^{-1}(\nabla \times E - K) \in H(\mathrm{curl};\Omega)$. Using (2.7) and the integration by parts formula (2.1), it follows that for all $F \in H_0(\mathrm{curl};\Omega)$

$$\begin{aligned}\int_\Omega ikJ \cdot \overline{F}\,\mathrm{d}x &= \int_\Omega \left(\nabla \times \left(\frac{1}{\mu}\left(\nabla \times E - K\right)\right)\right) \cdot \overline{F}\,\mathrm{d}x - \int_\Omega k^2\epsilon E \cdot \overline{F}\,\mathrm{d}x \\ &= \int_\Omega \left(\mu^{-1}\left(\nabla \times E - K\right)\right) \cdot \left(\nabla \times \overline{F}\right)\,\mathrm{d}x - \int_\Omega k^2\epsilon E \cdot \overline{F}\,\mathrm{d}x,\end{aligned}$$

and thus (2.6) holds.

On the other hand, if $E \in H(\mathrm{curl},\Omega)$ fulfills (2.6) for all $F \in H_0(\mathrm{curl},\Omega)$, then this also holds for all $F \in C_0^\infty(\Omega)^3$, which (by the definition of distributional derivatives) shows that

$$\nabla \times \left(\mu^{-1}\nabla \times E\right) - k^2\epsilon E = ikJ + \nabla \times \left(\mu^{-1}K\right),$$

and thus

$$\frac{1}{ik}\nabla \times \left(\mu^{-1}(\nabla \times E - K)\right) + ik\epsilon E = J.$$

Defining $H := \frac{1}{ik}\mu^{-1}(\nabla \times E - K)$ it follows that $H \in H(\mathrm{curl};\Omega)$ and that E and H solve (2.3)–(2.4).

The proof is complete. ∎

If $H(\mathrm{curl},\Omega)$ was compactly embedded into $L^2(\Omega)$, then Theorem 2.1 would immediately follow from Lemma 2.1 by a Fredholm argument. However this is not the case, and we need to introduce an additional variational formulation on the space

$$\mathcal{H} := \{E \in L^2(\Omega)^3 :\ \nabla \times E \in L^2(\Omega)^3,\ \nabla \cdot (\epsilon E) = 0,\ \nu \times E|_{\partial\Omega} = 0\},$$

which is compactly embedded into $L^2(\Omega)^3$ (see, e.g., [26, Theorem 5.32]). We now first consider the Maxwell system with homogeneous boundary data and divergence free electric currents, so that the solution lies in $\mathcal{H}$. After that we shall show that the general Maxwell system can be transformed (or gauged) to fulfill this condition.

Lemma 2.2. *For $f = 0$ and $\nabla \cdot J = 0$, $(E, H) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$ solves (2.3)–(2.5) if and only if $E \in \mathcal{H}$ solves*

$$\mathcal{B}(E, F) = \int_\Omega ikJ \cdot \overline{F}\,\mathrm{d}x + \int_\Omega \left(\mu^{-1}K\right) \cdot \left(\nabla \times \overline{F}\right)\,\mathrm{d}x \quad \text{for all } F \in \mathcal{H}, \tag{2.8}$$

and $H = -\frac{i}{k}\mu^{-1}(\nabla \times E - K)$.

Proof. If $(E, H) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$ fulfill (2.3)–(2.5), then clearly $E \in \mathcal{H}$ and Lemma 2.1 shows that (2.8) is fulfilled for all $F \in \mathcal{H} \subset H_0(\mathrm{curl}, \Omega)$.

To prove the other direction, let $E \in \mathcal{H}$ fulfill (2.8) for all $F \in \mathcal{H}$. Given $\Phi \in H_0(\mathrm{curl}, \Omega)$, there exists a solution $\varphi \in H_0^1(\Omega)$ of

$$\nabla \cdot (\epsilon \nabla \varphi) = -\nabla \cdot (\epsilon \Phi)$$

and thus $F := \Phi + \nabla\varphi \in \mathcal{H}$. Using (2.8) it follows that for

$$\begin{aligned}
\mathcal{B}(E, \Phi) &= \mathcal{B}(E, F) - \mathcal{B}(E, \nabla\varphi) \\
&= \int_\Omega ikJ \cdot \overline{F}\,\mathrm{d}x + \int_\Omega \left(\mu^{-1}K\right) \cdot \left(\nabla \times \overline{F}\right)\,\mathrm{d}x + \int_\Omega k^2 \epsilon E \cdot \nabla\overline{\varphi}\,\mathrm{d}x \\
&= \int_\Omega ikJ \cdot \overline{\Phi}\,\mathrm{d}x + \int_\Omega \left(\mu^{-1}K\right) \cdot \left(\nabla \times \overline{\Phi}\right)\,\mathrm{d}x,
\end{aligned}$$

where we used $\nabla \times (\nabla\varphi) = 0$, $\nabla \cdot J = 0$, $\nabla \cdot (\epsilon E) = 0$, and $\varphi|_{\partial\Omega} = 0$.

The proof is complete. ∎

We recall that we call $k > 0$ a resonant frequency, if the homogeneous Maxwell system (2.3)–(2.5) with $J = 0$, $K = 0$, and $f = 0$ admits a nontrivial solution.

Lemma 2.3. *If $k > 0$ is not a resonant frequency, then for every $J, K \in L^2(\Omega)^3$ with $\nabla \cdot J = 0$ and $f = 0$, there exists a unique solution $(E, H) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$ of (2.3)–(2.5), and the solution depends continuously on $J, K \in L^2(\Omega)^3$. Moreover, the set of resonant frequencies is discrete.*

Proof. Lemma 2.2 yields that $(E, H) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$ solves (2.3)–(2.5) if and only if

$$(\mathcal{A} + \mathcal{K}(k))\,E = l,$$

where $\mathcal{A}, \mathcal{K}(k) : \mathcal{H} \to \mathcal{H}^*$ are defined by

$$\begin{aligned}
\langle \mathcal{A}E, F\rangle &:= \int_\Omega \left(\mu^{-1}\nabla \times E\right) \cdot (\nabla \times F)\,\mathrm{d}x + \int_\Omega E \cdot F\,\mathrm{d}x && \text{for all } E, F \in \mathcal{H}, \\
\langle \mathcal{K}(k)E, F\rangle &:= -\int_\Omega (1 + k^2\epsilon)E \cdot F\,\mathrm{d}x && \text{for all } E, F \in \mathcal{H},
\end{aligned}$$

and $l \in \mathcal{H}^*$ is defined by

$$\langle l, F\rangle = \int_\Omega ikJ \cdot F\,\mathrm{d}x + \int_\Omega \left(\mu^{-1}K\right) \cdot (\nabla \times F)\,\mathrm{d}x \quad \text{for all } F \in \mathcal{H},$$

where $\mathcal{H}^*$ is the dual space of $\mathcal{H}$.

Then $\mathcal{A}$ is a coercive linear bounded operator and thus continuously invertible due to the Lax–Milgram theorem. For every $k \in \mathbb{C}$, $\mathcal{K}(k) : \mathcal{H} \to \mathcal{H}^*$ is a linear compact operator due to the compact embedding of $\mathcal{H}$ into $L^2(\Omega)^3$. $l \in \mathcal{H}^*$ depends linearly and continously on $J, K \in L^2(\Omega)^3$. It thus follows from the Fredholm alternative that $\mathcal{A} + \mathcal{K}(k)$ is continuously invertible if it is injective, i.e., if $k > 0$ is not a resonant frequency.

Moreover, $\mathcal{K}(k)$ depends analytically on k, and for $\widehat{k} := i$, $\mathcal{A}+\mathcal{K}(\widehat{k})$ is coercive and thus continuously invertible. Hence, it follows from the analytic Fredholm theorem that the set of resonances is discrete.

The proof is complete. ◻

Next we extend this result to nonhomogeneous boundary data f and non-divergence-free currents J and prove Theorem 2.1.

Proof of Theorem 2.1. Part (a) follows from Lemma 2.1. Part (b) and the uniqueness of the solution of the Maxwell system are proved in Lemma 2.3. To prove the existence of the solution, we let $J, K \in L^2(\Omega)^3$ and $f \in H^{-1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega)$. Define $E_f = \gamma_t^{-1} f \in H(\mathrm{curl}, \Omega)$, i.e., $\nu \times E_f|_{\partial\Omega} = f$ and E_f depends continuously and linearly on f. Moreover, we let $\psi \in H_0^1(\Omega)$ solve

$$\nabla \cdot (ik\epsilon\nabla\psi) = \nabla \cdot (J - ik\epsilon E_f),$$

which also depends continuously and linearly on J and E_f.

It follows from Lemma 2.3 that there exists a solution $(E_0, H) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$ of the gauged system

$$\begin{cases} \nabla \times E_0 - ik\mu H = K - \nabla \times E_f & \text{in } \Omega, \\ \nabla \times H + ik\epsilon E_0 = J - ik\epsilon E_f - ik\epsilon\nabla\psi & \text{in } \Omega, \\ \nu \times E_0|_{\partial\Omega} = 0, \end{cases}$$

and E_0 and H depends linearly and continuously on $K - \nabla \times E_f \in L^2(\Omega)^3$ and $J - ik\epsilon E_f - ik\epsilon\nabla\psi \in L^2(\Omega)^3$. Hence, $E := E_0 + E_f + \nabla\psi$ and H solve (2.3)–(2.5) and depend linearly and continuously on $J, K \in L^2(\Omega)^3$ and $f \in H^{-1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega)$.

The proof is complete. ◻

2.3. Unique continuation. The UCP is an important property to study the localized fields for differential equations. The UCP for the anisotropic Maxwell system was studied by [30, 36], which is of critical importance for our subsequent construction of the localized electromagnetic fields.

DEFINITION 2.1. *We say that (ϵ, μ) satisfies the UCP in Ω if any solution $(E, H) \in H(\mathrm{curl}; \Omega) \times H(\mathrm{curl}; \Omega)$ to the Maxwell system*

$$\begin{cases} \nabla \times E - ik\mu H = 0 & \text{in } \Omega, \\ \nabla \times H + ik\epsilon E = 0 & \text{in } \Omega \end{cases} \tag{2.9}$$

satisfies the property that if (E, H) vanishes in a nonempty open set D in Ω, then it must be identically vanishing in the whole domain Ω.

The UCP of the Maxwell system was proved by Leis [30] when the parameters ϵ, μ are C^2 scalar functions. When ϵ, μ are Lipschitz continuous anisotropic parameters, the UCP was proved by Nguyen and Wang [36]. In [32], Liu, Rondi, and Xiao have shown that the UCP holds for piecewise Lipschitz continuous matrix-valued functions ϵ and μ under some conditions, which we shall need for the subsequent study.

PROPOSITION 2.1 (UCP; Proposition 2.13 in [32]). *Given an open set $\widetilde{\Omega}$ in $\mathbb{R}^3$, let $\epsilon, \mu \in L^\infty(\widetilde{\Omega}, \mathbb{R}^{3\times 3})$ be matrix-valued functions in Ω satisfying* (1.1) *in $\widetilde{\Omega}$. Suppose the following:*

(1) *There is a family* $\{\Omega_i\}$ *of pairwise disjoint domains with* $\Omega_i \subset \widetilde{\Omega}$ *such that*

$$\widetilde{\Omega} \subset \cup_i \overline{\Omega_i}.$$

(2) *The set* $\Sigma_0 := \widetilde{\Omega} \cap (\cup_i \partial\Omega_i)$ *has Lebesuge measure zero (i.e.,* $|\Sigma_0| = 0$*).*

(3) *The point* $x \in \Sigma_0$ *is called to be a partition point is there exists* $\delta > 0$ *such that*

$$|B_\delta(x) \setminus (\Omega_i \cup \Omega_j)| = 0 \text{ for all } i \neq j$$

with $B_\delta(x) \cap \Omega_i$ *and* $B_\delta(x) \cap \Omega_j$ *being nonempty sets. Consider the set*

$$\mathcal{P}^c := \{x \in \Sigma_0 : \ x \text{ is not a partition point}\},$$

then we assume that $\widetilde{\Omega} \setminus \mathcal{P}^c$ *is connected.*

(4) *The functions* $(\epsilon, \mu) = (\epsilon_i, \mu_i)$ *in* Ω_i*, where* (ϵ_i, μ_i) *are locally Lipschitz matrix-valued functions in* $\widetilde{\Omega}$*.*

Then the UCP holds.

We have the following theorem.

Theorem 2.2. *Let* Ω *be a bounded Lipschitz domain and* $\mathcal{F} \subset \Omega$ *be a closed set in* $\mathbb{R}^3$ *such that* $\Omega \setminus \mathcal{F}$ *is connected to a relatively open boundary part* $\Gamma \subseteq \partial\Omega$*. Let* $(E, H) \in H(\mathrm{curl}; \Omega) \times H(\mathrm{curl}; \Omega)$ *solve*

$$\begin{cases} \nabla \times E - ik\mu H = 0 & \text{in } \Omega \backslash \mathcal{F}, \\ \nabla \times H + ik\epsilon E = 0 & \text{in } \Omega \backslash \mathcal{F}. \end{cases}$$

If $\nu \times E|_\Gamma = \nu \times H|_\Gamma = 0$ *on* Γ*, then* $(E, H) = (0, 0)$ *in* $\Omega \backslash \mathcal{F}$*.*

Proof. Let $\mathcal{O}$ be a nonempty open set in $\mathbb{R}^3$ such that $\mathcal{O} \cap \partial\Omega = \Gamma$ and $\mathcal{F} \subset \mathcal{O}$. In the open set $\widetilde{\Omega} := \Omega \cup \mathcal{O}$, we define

$$\widehat{\epsilon} := \begin{cases} \epsilon & \text{in } \Omega, \\ 1 & \text{in } \mathcal{O} \backslash \Omega, \end{cases} \text{ and } \widehat{\mu} = \begin{cases} \mu & \text{in } \Omega, \\ 1 & \text{in } \mathcal{O} \backslash \Omega. \end{cases}$$

It can be seen that the parameters $\widehat{\epsilon}$ and $\widehat{\mu}$ satisfy the conditions (1)–(4) in Proposition 2.1 in the open set $\widetilde{\Omega} = \Omega \cup \mathcal{O}$. Since $\nu \times E = \nu \times H = 0$ on Γ, we can extend (E, H) by $(0, 0)$ and define the extension functions

$$\widehat{E} := \begin{cases} E & \text{in } \Omega, \\ 0 & \text{in } \mathcal{O} \backslash \Omega, \end{cases} \text{ and } \widehat{H} := \begin{cases} H & \text{in } \Omega, \\ 0 & \text{in } \mathcal{O} \backslash \Omega. \end{cases}$$

First, we prove that $(\widehat{E}, \widehat{H}) \in H(\mathrm{curl}; \Omega \cup \mathcal{O}) \times H(\mathrm{curl}; \Omega \cup \mathcal{O})$. For any $\phi \in C_c^\infty(\Omega \cup \mathcal{O})$, we have

$$\begin{aligned} \int_{\Omega \cup \mathcal{O}} \widehat{E} \cdot (\nabla \times \phi) dx &= \int_\Omega E \cdot (\nabla \times \phi) dx \\ &= \int_\Omega (\nabla \times E) \cdot \phi dx + \int_{\partial\Omega} E \cdot (\nu \times \phi) dS \\ &= \int_{\Omega \cup \mathcal{O}} ((\nabla \times E) \, \chi_\Omega) \cdot \phi dx, \end{aligned}$$

where we have used $E \cdot (\nu \times \phi) = -\phi \cdot (\nu \times E) = 0$ on Γ and $\phi = 0$ on $\partial\Omega \backslash \Gamma$. This shows that $\widehat{E} \in H(\mathrm{curl}; \Omega \cup \mathcal{O})$ and that $\nabla \times \widehat{E}$ is the zero extension of $\nabla \times E$. The same holds for $\widehat{H}$, and thus it also follows that $(\widehat{E}, \widehat{H})$ is a solution of

$$\begin{cases} \nabla \times \widehat{E} - ik\widehat{\mu}\widehat{H} = 0 & \text{in } (\Omega \cup \mathcal{O}) \backslash \mathcal{F}, \\ \nabla \times \widehat{H} + ik\widehat{\epsilon}\widehat{E} = 0 & \text{in } (\Omega \cup \mathcal{O}) \backslash \mathcal{F}. \end{cases}$$

Notice that $\widehat{\epsilon}$ and $\widehat{\mu}$ are piecewise Lipschitz continuous functions fulfilling the ellipticity condition (1.1) and the conditions in Proposition 2.1. Recalling that $\widehat{E} = \widehat{H} = 0$ in $\mathcal{O} \backslash \overline{\Omega}$ (a nonempty open set), and by using Proposition 2.1, the UCP gives $\widehat{E} \equiv \widehat{H} \equiv 0$ in $(\Omega \cup \mathcal{O}) \backslash \mathcal{F}$.

The proof is complete. ∎

3. Localized electromagnetic fields. We are now in a position to present the main result on localizing and concentrating electromagnetic fields. We show that there exists boundary data (supported on an arbitrarily small boundary part) which can generate an electromagnetic field with an arbitrarily high energy on one part of the considered domain and an arbitrarily small energy on another part. This extends the related results in [12] for the conductivity equation and [20] for the Helmholtz equation to the more practical and challenging Maxwell system. In this section, we prove the existence of localized fields by using the functional analysis techniques from [12]. Recall our main result as follows.

THEOREM 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $\Gamma \subseteq \partial\Omega$ be a relatively open piece of the boundary. Let $\epsilon, \mu \in L^\infty(\Omega, \mathbb{R}^{3\times 3})$ be real-valued, piecewise Lipschitz continuous functions satisfying* (1.1) *and $k \in \mathbb{R}_+$ be a nonresonant frequency. Let $D \Subset \Omega$ be a closed set with a connected complement $\Omega \setminus D$. For every open set $M \subseteq \Omega$ with $M \not\subseteq D$ (see Figure* 1 *for the schematic illustration), there exists a sequence $\left\{f^{(\ell)}\right\}_{\ell\in\mathbb{N}} \subset C_c^\infty(\Gamma)$ such that the electromagnetic fields fulfill*

(3.1)
$$\int_M \left(|E^{(\ell)}|^2 + |H^{(\ell)}|^2\right) \mathrm{d}x \to \infty \quad \text{and} \quad \int_D \left(|E^{(\ell)}|^2 + |H^{(\ell)}|^2\right) \mathrm{d}x \to 0 \text{ as } \ell \to \infty,$$

where, for $\ell \in \mathbb{N}$, $(E^{(\ell)}, H^{(\ell)}) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$ is a solution of

$$\nabla \times E^{(\ell)} - ik\mu H^{(\ell)} = 0 \quad \text{in } \Omega, \tag{3.2}$$

$$\nabla \times H^{(\ell)} + ik\epsilon E^{(\ell)} = 0 \quad \text{in } \Omega \tag{3.3}$$

with the boundary data

$$\nu \times E^{(\ell)}|_{\partial\Omega} = \begin{cases} f^{(\ell)} & \text{on } \Gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Proof of Theorem 3.1. We first note that it suffices to prove the theorem for an open subset of M. Hence, without loss of generality, we can assume that $M \Subset \Omega$ is open, $\overline{M} \cap D = \emptyset$, and $\Omega \setminus (\overline{M} \cup D)$ is connected. We follow the localized potentials strategy in [12, 20, 21] and first describe the energy terms in Theorem 3.1 as operator evaluations. Then we show that the ranges of the adjoints of these operators have trivial intersection. A functional analytic relation between the norm of an operator evaluation and the range of its adjoint then yields that the operator evaluations cannot be bounded by each other, which then shows that we can drive one energy term in Theorem 3.1 to infinity and the other one to zero.

For a measurable subset $O \subseteq \Omega$, we define

$$\mathcal{L}_O : \ H(\Gamma) \to L^2(O)^3 \times L^2(O)^3 \text{ by } f \mapsto (E|_O, H|_O),$$

where $H(\Gamma)$ is defined in (2.2), and $(E, H) \in L^2(\Omega)^3 \times L^2(\Omega)^3$ solves (3.2)–(3.3) with the boundary data $\nu \times E|_{\partial\Omega} = f$. Now we characterize the adjoint of this operator.

LEMMA 3.1. *The adjoint of $\mathcal{L}_O$ is given by*

$$\mathcal{L}_O^* : \ L^2(O)^3 \times L^2(O)^3 \to H(\Gamma)^* \ \textit{by } (J, K) \to -\nu \times (\widetilde{H} \times \nu)|_\Gamma,$$

where $(\widetilde{E}, \widetilde{F}) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$ solves the (adjoint) Maxwell system (cf. Theorem 2.1)

$$\begin{cases} \nabla \times \widetilde{E} + ik\mu\widetilde{H} = K\chi_O & \text{in } \Omega, \\ \nabla \times \widetilde{H} - ik\epsilon\widetilde{E} = J\chi_O & \text{in } \Omega, \\ \nu \times \widetilde{E}|_{\partial\Omega} = 0, \end{cases}$$

and $K\chi_O$ and $J\chi_O$ denote the zero extensions of K and J to Ω, respectively.

Proof. Similar to section 2.1, we write the dual pairing on $H(\Gamma)^* \times H(\Gamma)$ as an integral for the sake of notational convenience. With this notation we have that

$$\begin{aligned}
&\int_\Gamma \overline{f} \cdot \mathcal{L}_O^*(J, K)\, \mathrm{d}s \\
&= \int_O (J, K) \cdot \overline{(E|_O, H|_O)}\, \mathrm{d}x = \int_\Omega J\chi_O \cdot \overline{E}\, \mathrm{d}x + \int_\Omega K\chi_O \cdot \overline{H}\, \mathrm{d}x \\
&= \int_\Omega J\chi_O \cdot \overline{E}\, \mathrm{d}x - \frac{1}{ik} \int_\Omega K\chi_O \cdot \left(\mu^{-1}\nabla \times \overline{E}\right) \mathrm{d}x \\
&= \int_\Omega \left(\nabla \times \widetilde{H} - ik\epsilon\widetilde{E}\right) \cdot \overline{E}\, \mathrm{d}x - \frac{1}{ik} \int_\Omega \left(\nabla \times \widetilde{E} + ik\mu\widetilde{H}\right) \cdot \left(\mu^{-1}\nabla \times \overline{E}\right) \mathrm{d}x \\
&= \int_\Omega \left(\nabla \times \widetilde{H}\right) \cdot \overline{E}\, \mathrm{d}x - \int_\Omega \widetilde{H} \cdot \left(\nabla \times \overline{E}\right) \mathrm{d}x \\
&\quad - \frac{1}{ik}\left(\int_\Omega \left(\mu^{-1}\nabla \times \overline{E}\right) \cdot \left(\nabla \times \widetilde{E}\right) \mathrm{d}x - \int_\Omega k^2\epsilon\overline{E} \cdot \widetilde{E}\, \mathrm{d}x\right) \\
&= -\int_{\partial\Omega} \left(\nu \times \overline{E}|_{\partial\Omega}\right) \cdot \left(\nu \times (\widetilde{H}|_{\partial\Omega} \times \nu)\right) \mathrm{d}S = -\int_\Gamma \overline{f} \cdot \left(\nu \times (\widetilde{H}|_\Gamma \times \nu)\right) \mathrm{d}S,
\end{aligned}$$

where we make use of the fact that ϵ and μ are real-valued and symmetric. We also utilized the integration by parts formula (2.1) and that $\widetilde{E} \in H_0(\mathrm{curl}, \Omega)$ implies that

$$\int_\Omega \left(\mu^{-1}\nabla \times \overline{E}\right) \cdot \left(\nabla \times \widetilde{E}\right) \mathrm{d}x - \int_\Omega k^2\epsilon\overline{E} \cdot \widetilde{E}\, \mathrm{d}x = 0$$

by Theorem 2.1(a). ∎

Next we show the following property for the ranges of the adjoint operators $\mathcal{L}_M^*$ and $\mathcal{L}_D^*$.

LEMMA 3.2. *$\mathcal{L}_M$ and $\mathcal{L}_D^*$ are injective, the ranges $\mathscr{R}(\mathcal{L}_M^*)$ and $\mathscr{R}(\mathcal{L}_D^*)$ are both dense in $H(\Gamma)^*$, and*

$$\mathscr{R}(\mathcal{L}_M^*) \cap \mathscr{R}(\mathcal{L}_D^*) = \{0\}. \tag{3.5}$$

Proof. The proof follows from the UCP for the Maxwell system. By Proposition 2.1, one readily sees that $\mathcal{L}_M$ and $\mathcal{L}_D$ are injective, and therefore $\mathscr{R}(\mathcal{L}_M^*)$ and $\mathscr{R}(\mathcal{L}_D^*)$ both are dense in $H(\Gamma)^*$.

To prove (3.5), let $g \in \mathscr{R}(\mathcal{L}_M^*) \cap \mathscr{R}(\mathcal{L}_D^*)$, and then there exist $J_M, K_M \in L^2(M)$ and $J_D, K_D \in L^2(D)$ such that the solutions $(E_M, H_M), (E_D, H_D) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$ of

$$\begin{cases} \nabla \times E_M + ik\mu H_M = K_M \chi_M & \text{in } \Omega, \\ \nabla \times H_M - ik\epsilon E_M = J_M \chi_M & \text{in } \Omega, \\ \nu \times E_M|_{\partial\Omega} = 0, & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \nabla \times E_D + ik\mu H_D = K_D \chi_D & \text{in } \Omega, \\ \nabla \times H_D - ik\epsilon E_D = J_D \chi_D & \text{in } \Omega, \\ \nu \times E_D|_{\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases}$$

fulfill

$$\nu \times (H_M \times \nu)|_\Gamma = g = \nu \times (H_D \times \nu)|_\Gamma.$$

Since $\Omega \setminus (\overline{M} \cup D)$ is connected, we obtain by using Theorem 2.2 that

$$E_M = E_D \text{ in } \Omega \setminus (\overline{M} \cup D).$$

Hence, we can define

$$\mathbb{E} := \begin{cases} E_D & \text{in } M, \\ E_M & \text{in } D, \\ E_D = E_M & \text{in } \Omega \setminus (\overline{M} \cup D), \end{cases} \quad \text{and} \quad \mathbb{H} := \begin{cases} H_D & \text{in } M, \\ H_M & \text{in } D, \\ H_D = H_M & \text{in } \Omega \setminus (\overline{M} \cup D). \end{cases}$$

As in the proof of UCP, Theorem 2.2, it is easy to see that $(\mathbb{E}, \mathbb{H}) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$ is a solution of

$$\begin{cases} \nabla \times \mathbb{E} + ik\mu \mathbb{H} = 0 & \text{in } \Omega, \\ \nabla \times \mathbb{H} - ik\epsilon \mathbb{E} = 0 & \text{in } \Omega, \\ \nu \times \mathbb{E}|_{\partial\Omega} = 0 & \text{on } \partial\Omega. \end{cases} \qquad \square$$

Since k is nonresonant, it follows that $(\mathbb{E}, \mathbb{H}) = (0, 0)$, and hence $g = 0$. This completes the proof.

Now we can use the following tool from functional analysis.

LEMMA 3.3. *Let X, Y_1, and Y_2 be Hilbert spaces, and $\mathcal{A}_1 : X \to Y_1$ and $\mathcal{A}_2 : X \to Y_2$ be linear bounded operators. Then*

$$\exists C > 0 : \ \|\mathcal{A}_1 x\| \leq C \|\mathcal{A}_2 x\| \quad \textit{for all } x \in X \quad \textit{if and only if} \quad \mathscr{R}(\mathcal{A}_1^*) \subseteq \mathscr{R}(\mathcal{A}_2^*).$$

Proof. This is proved for reflexive Banach spaces in [12, Lemma 2.5]. $\square$

Proof of Theorem 1.1. From Lemma 3.2 it follows that $\mathscr{R}(\mathcal{L}_M^*) \not\subseteq \mathscr{R}(\mathcal{L}_D^*)$. Using Lemma 3.3 this shows that

$$\nexists C > 0 : \ \|\mathcal{L}_M f\| \leq C \|\mathcal{L}_D f\| \quad \text{for all } f \in H(\Gamma),$$

and by continuity of $\mathcal{L}_M$ and $\mathcal{L}_D$ and density of $C_c^\infty(\Gamma) \subset H(\Gamma)$ this is equivalent to

$$\nexists C > 0 : \ \|\mathcal{L}_M f\| \leq C \|\mathcal{L}_D f\| \quad \text{for all } f \in C_c^\infty(\Gamma). \tag{3.6}$$

Using (3.6) with $C := \ell^2$ for all $\ell \in \mathbb{N}$ we thus obtain a sequence $\left\{ \widetilde{f}^{(\ell)} \right\}_{\ell \in \mathbb{N}} \subset C_c^\infty(\Gamma)$ with

$$\|\mathcal{L}_M \widetilde{f}^{(\ell)}\| > \ell^2 \|\mathcal{L}_D \widetilde{f}^{(\ell)}\| \quad \text{for all } \ell \in \mathbb{N}.$$

By injectivity $\mathcal{L}_M \widetilde{f}^{(\ell)} \neq 0$ implies $\widetilde{f}^{(\ell)} \neq 0$ and $\mathcal{L}_D \widetilde{f}^{(\ell)} \neq 0$, so that we can define

$$f^{(\ell)} := \frac{\widetilde{f}^{(\ell)}}{\ell \|\mathcal{L}_D \widetilde{f}^{(\ell)}\|} \in C_c^\infty(\Gamma)$$

and it follows that

$$\int_M \left(|E^{(\ell)}|^2 + |H^{(\ell)}|^2 \right) \mathrm{d}x = \|\mathcal{L}_M f^{(\ell)}\|^2 > \ell^2 \to \infty,$$
$$\int_D \left(|E^{(\ell)}|^2 + |H^{(\ell)}|^2 \right) \mathrm{d}x = \|\mathcal{L}_D f^{(\ell)}\|^2 = \frac{1}{\ell^2} \to 0.$$

This completes the proof of Theorem 1.1. ∎

Remark 3.1.
(a) A constructive version of the existence proof for the localized fields can be obtained as in [12, Lemma 2.8].
(b) With the same arguments as in [20, section 4.1] one can also show that for all spaces $W \subseteq H(\Gamma)$ with finite codimensions, one can find a sequence $\left\{f^{(\ell)}\right\}_{\ell\in\mathbb{N}} \subset W$ such that the corresponding electromagnetic fields fulfill (3.1). This might be useful for developing monotonicity-based reconstruction methods as in [20].

4. Runge approximation property for the partial data Maxwell system. In this section we derive an extension of the localization result in Theorem 1.1 and establish a *Runge approximation property* for the partial data Maxwell system, which is of mathematical interest for its own sake. We show that every solution of the Maxwell system on a subset of Ω with a Lipschitz boundary and a connected complement can be approximated arbitrarily well by a sequence of solutions on the whole domain Ω with partial boundary data. Since we can choose a solution that is zero on a part of Ω and nonzero on another part of Ω, this readily implies a fortiori the localization result in Theorem 1.1. We also refer to [20] for the connection between Runge approximation properties and localized solutions.

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $\Gamma \subseteq \partial\Omega$ be a relatively open piece of the boundary. Let $\epsilon, \mu \in L^\infty(\Omega, \mathbb{R}^{3\times 3})$ be real-valued, piecewise Lipschitz continuous functions satisfying* (1.1) *and $k \in \mathbb{R}_+$ be a nonresonant frequency.*

Let $O \Subset \Omega$ be an open set with Lipschitz boundary and connected complement $\Omega \setminus \overline{O}$. For every solution $(e, h) \in H(\mathrm{curl}, O) \times H(\mathrm{curl}, O)$ of

$$\nabla \times e - ik\mu h = 0 \quad \text{in } O, \tag{4.1}$$
$$\nabla \times h + ik\epsilon e = 0 \quad \text{in } O, \tag{4.2}$$

there exists a sequence $\left\{f^{(\ell)}\right\}_{\ell\in\mathbb{N}} \subset C_c^\infty(\Gamma)$ such that the electromagnetic fields fulfill

$$\|E^{(\ell)} - e\|_{L^2(O)} \to 0 \quad \text{and} \quad \|H^{(\ell)} - h\|_{L^2(O)} \to 0 \quad \text{as} \quad \ell \to \infty,$$

where $(E^{(\ell)}, H^{(\ell)}) \in H(\mathrm{curl}, O) \times H(\mathrm{curl}, O)$ solve (3.2)–(3.4) *in O.*

Proof. Let $(e, h) \in H(\mathrm{curl}, O) \times H(\mathrm{curl}, O)$ solve (4.1)–(4.2). With the operator $\mathcal{L}_O$ introduced in section 3, we shall show that

$$(e, h) \in \overline{\mathscr{R}(\mathcal{L}_O)} = \left(\mathscr{R}(\mathcal{L}_O)^\perp\right)^\perp = \mathscr{N}(\mathcal{L}_O^*)^\perp, \tag{4.3}$$

where the closure and the orthogonal complement are understood with respect to the $L^2(O)^3 \times L^2(O)^3$-scalar product. This shows that the assertion holds with a sequence $\{f^{(\ell)}\}_{\ell\in\mathbb{N}} \subset H(\Gamma)$, and it follows by density that the assertion holds with a sequence $\{f^{(\ell)}\}_{\ell\in\mathbb{N}} \subset C_c^\infty(\Gamma)$.

To prove (4.3), we let $(J,K) \in \mathcal{N}(\mathcal{L}_O^*) \subseteq L^2(O)^3 \times L^2(O)^3$. Then, by Lemma 3.1, there exists $(E,H) \in H(\mathrm{curl}, O) \times H(\mathrm{curl}, O)$ that solves

$$\begin{cases} \nabla \times E + ik\mu H = K\chi_O & \text{in } \Omega, \\ \nabla \times H - ik\epsilon E = J\chi_O & \text{in } \Omega, \\ \nu \times E|_{\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases}$$

with $\nu \times (H \times \nu)|_\Gamma = \mathcal{L}_O^*(J,K) = 0$. The UCP in Theorem 2.2 implies that $(E,H) = (0,0)$ on $\Omega \setminus \overline{O}$ and thus

$$\nu \times E|_{\partial O} = 0 = \nu \times H|_{\partial O}.$$

Hence, using the integration by parts formula (2.1), it follows that

$$\begin{aligned} &\int_O \left(e \cdot \overline{J} + h \cdot \overline{K}\right) \mathrm{d}x \\ &\quad= \int_O e \cdot \left(\nabla \times \overline{H} + ik\epsilon\overline{E}\right) \mathrm{d}x + \int_O h \cdot \left(\nabla \times \overline{E} - ik\mu\overline{H}\right) \mathrm{d}x \\ &\quad= \int_O \left((\nabla \times e) \cdot \overline{H} + ik\epsilon e \cdot \overline{E}\right) \mathrm{d}x + \int_O \left((\nabla \times h) \cdot \overline{E} - ik\mu h \cdot \overline{H}\right) \mathrm{d}x \\ &\quad= \int_O (\nabla \times e - ik\mu h) \cdot \overline{H}\, \mathrm{d}x + \int_O (\nabla \times h + ik\epsilon e) \cdot \overline{E}\, \mathrm{d}x = 0. \end{aligned}$$

This shows $(e,h) \perp (J,K)$ so that (4.3) holds, and thus the assertion is proved. □

Remark 4.1. The Runge approximation property in Theorem 4.1 implies the localization property in Theorem 1.1 by the following argument. Let $D \Subset \Omega$ be a closed set with a connected complement $\Omega \setminus D$, and $M \subseteq \Omega$ be an open set with $M \not\subseteq D$ (see again Figure 1). By shrinking M and enlarging D, we can assume that M is an open set and D is a closed set with Lipschitz boundaries, $\overline{M} \cup D \Subset \Omega$, and $\overline{M} \cap D = \emptyset$ and that $\Omega \setminus (\overline{M} \cup D)$ is connected.

The UCP in Theorem 2.2 implies that a solution of the Maxwell system in Ω with nontrivial boundary data cannot vanish identically on M. This shows that there exists a nonzero solution of the Maxwell system on M. We extend this solution by zero on D and obtain a solution $(e,h) \in H(\mathrm{curl}, O) \times H(\mathrm{curl}, O)$ on $O := M \cup \mathrm{int}D$ with $(e,h)|_{\mathrm{int}D} \equiv (0,0)$ and $(e,h)|_M \not\equiv (0,0)$. Then the Runge approximation sequence from Theorem 4.1 converges to zero on D but not on M and a simple scaling argument as in the proof of Theorem 1.1 in section 3 gives a sequence of electromagnetic fields such that

$$\int_M \left(|E^{(\ell)}|^2 + |H^{(\ell)}|^2\right) \mathrm{d}x \to \infty \quad \text{and} \quad \int_D \left(|E^{(\ell)}|^2 + |H^{(\ell)}|^2\right) \mathrm{d}x \to 0 \text{ as } \ell \to \infty,$$

and $\nu \times E^{(\ell)} \in C_c^\infty(\Gamma)$ for all $\ell \in \mathbb{N}$, which also proves Theorem 1.1.

5. Concluding remarks. We considered field localizing and concentrating for the electromagnetic waves governed by the time-harmonic Maxwell system in a bounded domain occupied by a given medium that is generic and could be anisotropic. It has been shown that through proper boundary inputs, one can generate electromagnetic fields with the corresponding energy concentrated in a given subregion while nearly vanishing in another given subregion. We would like to emphasize that the localizing results are known previously for several scalar models including the conductivity equation and the Helmholtz equation, but the extension to the Maxwell system with nonsmooth anisotropic coefficients requires considerable care and technical involvement. As pointed out in Remark 3.1, the result can be used to develop monotonicity-based reconstruction methods for inverse problems associated with the electromagnetic waves. In fact, with the localizing result established in the present article, this can be done by following a spirit similar to the one in [20] for the Helmholtz equation by one of the authors of this article. We also propose more applications of practical interest, including telecommunications and inductive charging. This new perspective poses interesting problems for future investigation. For example, for certain given boundary inputs, one may consider to construct a specific medium configuration for localizing and concentrating the electromagnetic fields in a desirable way.

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