LOCAL UNIQUENESS FOR AN INVERSE BOUNDARY VALUE PROBLEM WITH PARTIAL DATA

BASTIAN HARRACH AND MARCEL ULLRICH

Abstract. In dimension $n \geq 3$, we prove a local uniqueness result for the potentials q of the Schrödinger equation $-\Delta u + qu = 0$ from partial boundary data. More precisely, we show that potentials $q_1, q_2 \in L^\infty$ with positive essential infima can be distinguished by local boundary data if there is a neighborhood of a boundary part where $q_1 \geq q_2$ and $q_1 \not\equiv q_2$.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, be a Lipschitz-domain with outer normal ν and $L^\infty_+(\Omega)$ denote the subspace of $L^\infty(\Omega)$-functions with positive essential infima. We consider the question whether the potential $q \in L^\infty_+(\Omega)$ in the Schrödinger equation

$$-\Delta u + qu = 0 \quad \text{in} \quad \Omega \tag{1.1}$$

is uniquely determined by partial boundary data on a possibly arbitrarily small non-empty relatively open subset $\Gamma \subseteq \partial\Omega$.

For such a boundary subset $\Gamma \subseteq \partial\Omega$ and $q \in L^\infty_+(\Omega)$, the partial boundary data, we consider in this work, is given by the *local Neumann-to-Dirichlet (NtD) operator*

$$\Lambda_\Gamma(q) \,:\, L^2(\Gamma) \to L^2(\Gamma), \quad g \mapsto u_q^{(g)}|_\Gamma, \tag{1.2}$$

where $u_q^{(g)} \in H^1(\Omega)$ is the solution of

$$-\Delta u_q^{(g)} + q u_q^{(g)} = 0 \text{ in } \Omega \qquad \text{with} \qquad \partial_\nu u_q^{(g)}|_{\partial\Omega} = \begin{cases} g, & \text{on } \Gamma, \\ 0, & \text{on } \partial\Omega \setminus \Gamma. \end{cases} \tag{1.3}$$

$\Lambda_\Gamma(q)$ is easily shown to be a compact self-adjoint linear operator.

In this article, we will show the following local uniqueness result.

Theorem 1.1. *Let $q_1, q_2 \in L^\infty_+(\Omega)$ and $V \subseteq \mathbb{R}^n$ be an open connected set with $q_1 \geq q_2$ on $\Omega \cap V$ and $\Gamma := \partial\Omega \cap V \neq \emptyset$. Then,*

$$q_1|_{\Omega\cap V} \not\equiv q_2|_{\Omega\cap V} \quad \textit{implies} \quad \Lambda_\Gamma(q_1) \neq \Lambda_\Gamma(q_2). \tag{1.4}$$

Moreover, in that case $\Lambda_\Gamma(q_2) - \Lambda_\Gamma(q_1)$ has a positive eigenvalue.

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The *inverse potential problem* of the Schrödinger equation is closely related to the *inverse conductivity problem (Calderón Problem* [4, 5]*)*. For both problems, uniqueness from full boundary data on $\partial\Omega$ has been extensively studied in the last 30 years. To give a brief overview of prominent contributions, we list Kohn and Vogelius [22, 23], Sylvester and Uhlmann [27], Nachman [25], Astala and Päivärinta [1], Bukhgeim [3], Haberman and Tataru [9].

The uniqueness problem from partial boundary data has attracted growing attention over the last years. Typically, this is studied for data of type C_q^D or C_q^N on sets $\Gamma^D, \Gamma^N \subseteq \partial\Omega$, where

$$C_q^D := \left\{ (u|_{\Gamma^D}, \partial_\nu u|_{\Gamma^N}) \,:\, -\Delta u + qu = 0,\ \mathrm{supp}\,(u|_{\partial\Omega}) \subseteq \Gamma^D \right\},$$

$$C_q^N := \left\{ (u|_{\Gamma^D}, \partial_\nu u|_{\Gamma^N}) \,:\, -\Delta u + qu = 0,\ \mathrm{supp}\,(\partial_\nu u|_{\partial\Omega}) \subseteq \Gamma^N \right\}.$$

Obviously, for potentials $q \in L_+^\infty(\Omega)$, the question of uniqueness from data of type C_q^N with $\Gamma = \Gamma^D = \Gamma^N$ is equivalent to the question of uniqueness from the local NtD operator $\Lambda_\Gamma(q)$.

Hereafter, we list some recent results. Let us also refer to the article of Kenig and Salo [21] for a further overview.

For dimension $n = 2$, Imanuvilov, Uhlmann and Yamamoto showed uniqueness from data of type C_q^D in [16], where $\Gamma^D = \Gamma^N$ is an arbitrary open subset in $\partial\Omega$ and the potentials are in $\mathcal{C}^{2+\alpha}\left(\overline{\Omega}\right)$ for $\alpha > 0$.

For dimension $n \geq 3$, Kenig, Sjöstrand and Uhlmann proved uniqueness from data of type C_q^D for $q \in L^\infty(\Omega)$ in [20], where Γ^D and Γ^N are open neighborhoods slightly larger than a front face and a back face of $\partial\Omega$, respectively. Nachman and Street presented a constructive proof of this result in [24]. In [17], Isakov proved uniqueness from data of type C_q^D for $q \in L^\infty(\Omega)$ assuming $\Gamma^D = \Gamma^N$ and that the remaining boundary part is contained in a plane or a sphere. In [19], Kenig and Salo presented a result that unifies and improves the approaches of [20] and [17]. In particular, they reduced the assumptions regarding the sets Γ^D, Γ^N and, if so (in [17]), the remaining boundary part. Let us refer to their article for a detailed description.

Theorem 1.1 is, to the knowledge of the authors, the first result that presents a uniqueness result for partial data on an arbitrary non-empty relatively open boundary part $\Gamma \subseteq \partial\Omega$ (with $\Gamma = \Gamma^D = \Gamma^N$) for dimension $n \geq 3$. Except the assumption that Ω has to be a Lipschitz-domain, there are no further assumptions to the boundary required: neither to the boundary part Γ nor to the remaining boundary part.

This paper is organized as follows. In Section 2, we prove Theorem 1.1. For this purpose, we present and combine a *monotonicity relation* for the local NtD operator (Lemma 2.1) and a new variant of the concept of *localized potentials* (Lemma 2.2, cf. [8] for the initial concept). Lemma 2.1 presents a monotonicity inequality that yields a lower bound for the change of the local NtD operator (represented by its corresponding quadratic form) caused by a potential change. This lower bound depends on the spatial change of the potential weighted by the solution of the Schrödinger equation for the initial potential. Lemma 2.2 shows a possibility to control the lower bound of the monotonicity inequality.[1] The approach of combining a monotonicity relation with the concept of localized potentials has previously been used in [13, 11, 12]. The proofs of the Lemmas 2.1 and 2.2 are postponed to the Sections 3 and 4, respectively.

2. The proof of Theorem 1.1

Let $q_1, q_2 \in L^\infty_+(\Omega)$, let $V \subseteq \mathbb{R}^n$ be an open connected set and let $\Gamma := \partial\Omega \cap V \neq \emptyset$.

To prove Theorem 1.1, we combine a monotonicity inequality (Lemma 2.1) for Neumann-to-Dirichlet operators with a result about the existence of localized potentials (Lemma 2.2).

Lemma 2.1. *Let $g \in L^2(\Gamma)$ and $u_1 := u_{q_1}^{(g)} \in H^1(\Omega)$ be the corresponding solution of (1.3). Then,*

$$(g, (\Lambda_\Gamma(q_2) - \Lambda_\Gamma(q_1))\, g)_{L^2(\Gamma)} \geq -\int_\Omega (q_2 - q_1){u_1}^2 \, \mathrm{d}x. \tag{2.1}$$

Lemma 2.1 is proven in Section 3.

Lemma 2.2. *Let $q_1 \gneq q_2$ on $\Omega \cap V$ (i.e., $q_1|_{\Omega\cap V} \geq q_2|_{\Omega\cap V}$ and $q_1|_{\Omega\cap V} \not\equiv q_2|_{\Omega\cap V}$). Then, there exists a sequence $(g_m)_{m\in\mathbb{N}} \subset L^2(\Gamma)$ such that the corresponding solutions $(u_m)_{m\in\mathbb{N}} := \left(u_{q_1}^{(g_m)}\right)_{m\in\mathbb{N}} \subset H^1(\Omega)$ of (1.3) fulfill*

$$\lim_{m\to\infty} \int_{V\cap\Omega} (q_1 - q_2){u_m}^2 \, \mathrm{d}x = \infty \quad \text{and} \quad \lim_{m\to\infty} \int_{\Omega\setminus V} (q_1 - q_2){u_m}^2 \, \mathrm{d}x = 0. \tag{2.2}$$

Lemma 2.2 is proven in Section 4.

Proof of Theorem 1.1. First, we apply Lemma 2.2: There exists a $g \in L^2(\Gamma)$ such that the corresponding solution $u := u_{q_1}^{(g)}$ of (1.3) fulfills

$$\int_{V\cap\Omega} (q_1 - q_2)u^2 \, \mathrm{d}x > 1 \quad \text{and} \quad \int_{\Omega\setminus V} (q_1 - q_2)u^2 \, \mathrm{d}x > -1.$$

Now, we apply Lemma 2.1 and obtain

$$\begin{aligned}
(g, (\Lambda_\Gamma(q_2) - \Lambda_\Gamma(q_1))\, g)_{L^2(\Gamma)} &\geq -\int_\Omega (q_2 - q_1)u^2 \, \mathrm{d}x \\
&= \int_{V\cap\Omega} (q_1 - q_2)u^2 \, \mathrm{d}x + \int_{\Omega\setminus V} (q_1 - q_2)u^2 \, \mathrm{d}x \\
&> 1 - 1 = 0.
\end{aligned}$$

This shows that $\Lambda_\Gamma(q_2) - \Lambda_\Gamma(q_1)$ is not semi negative definite and thus has a positive eigenvalue. □

[1]Originally, the concept of localized potentials was used to locally control electrical potentials for the inverse conductivity problem. Since in this work it is used to locally weight the potentials of the Schrödinger equation, it seems appropriate to keep with the name “localized potentials”.

3. Monotonicity for Neumann-to-Dirichlet maps

Again, let $q_1, q_2 \in L^\infty_+(\Omega)$, $V \subseteq \mathbb{R}^n$ be an open connected set and $\Gamma := \partial\Omega \cap V \neq \emptyset$.

Such monotonicity estimates are well-known for the inverse conductivity problem, cf., e.g., Ikehata, Kang, Seo, and Sheen [18, 15].

Lemma 2.1 follows from [11, Lemma 4.1]. Since the proof is simple and short, we include it for the sake of completeness.

Proof of Lemma 2.1. Let $g \in L^2(\Gamma)$ and $u_i := u_{q_i}^{(g)} \in H^1(\Omega)$ be the corresponding solutions of (1.3) for $i \in \{1, 2\}$. Then,

$$b_i(u_i, w) := \int_\Omega \nabla u_i \nabla w + q_i u w \,\mathrm{d}x = \int_\Gamma g w|_\Gamma \,\mathrm{d}s =: l(w) \quad \forall w \in H^1(\Omega),\ i \in \{1, 2\}.$$

Now, we use this and consider

$$\begin{aligned}
&(g, (\Lambda_\Gamma(q_2) - \Lambda_\Gamma(q_1))\, g)_{L^2(\Gamma)} \\
&\quad = l(u_2) - l(u_1) = b_2(u_2, u_2) - 2b_2(u_2, u_1) + b_1(u_1, u_1) \\
&\quad = -\int_\Omega (q_2 - q_1) u_1^2 - (\nabla(u_2 - u_1))^2 - q_2(u_1 - u_2)^2 \,\mathrm{d}x.
\end{aligned}$$

Since $q_2 \geq 0$, the assertion follows. $\square$

4. Localized Potentials

Again, let $q_1, q_2 \in L^\infty_+(\Omega)$, $V \subseteq \mathbb{R}^n$ be an open connected set and $\Gamma := \partial\Omega \cap V \neq \emptyset$. In addition, as assumed in Lemma 2.2, let $q_1 \gneq q_2$ on $\Omega \cap V$ (i.e., $q_1|_{\Omega\cap V} \geq q_2|_{\Omega\cap V}$ and $q_1|_{\Omega\cap V} \not\equiv q_2|_{\Omega\cap V}$).

Since the open set $V \cap \Omega$ is a countable union of closed balls and $q_1 \gneq q_2$ on $\Omega \cap V$, there exists a closed ball

$$B \subseteq V \cap \Omega \quad \text{where} \quad q_1 \gneq q_2 \tag{4.1}$$

and $V \setminus B$ is connected.

To prove Lemma 2.2, we introduce two operators in Definition 4.3 and present some properties of these operators and their adjoints in Lemma 4.4. In the proof of Lemma 4.4, the following two theorems play a key role.

Theorem 4.1. *Let H_1, H_2 be two Hilbert spaces, $L \in \mathcal{L}(H_1, H_2)$ and $h \in H_2$. Then,*

$$h \in \mathcal{R}(L) \quad \Leftrightarrow \quad \exists C > 0 : |(h, g)_{H_2}| \leq C \|L^* g\|_{H_1} \quad \forall g \in H_2. \tag{4.2}$$

Proof. This is a well-known result from functional analysis (see, e.g., the book of Bourbarki [2]). For Banach spaces, a proof is given in [7, Lemma 3.4]. $\square$

Theorem 4.2 (Unique continuation from sets of positive measure)**.** *Let $\Omega' \in \mathbb{R}^m$, $m \geq 3$, be a connected open set and $q \in L^\infty(\Omega')$. The trivial solution of*

$$-\Delta u + qu = 0 \tag{4.3}$$

is the only $H^1(\Omega')$-solution vanishing on a measurable set of positive measure.

Proof. Theorem 4.2 is the combination of the following two results (cf. the work of Rachid Regbaoui [26, proof of Theorem 2.1]).

(a) $H^1(\Omega')$-solutions of (4.3) that vanish on a measurable set of positive measure have zeros of infinite order (see, e.g., the result of de Figueiredo and Gossez [6, Proposition 3] or the result of Hadi and Tsouli [10, Theorem 2.1]).
(b) The trivial solution $u = 0$ is the only $H^1(\Omega')$-solution of (4.3) that has a zero of infinite order (see, e.g, the book of Hörmander [14, Theorem 17.2.6]).

□

Definition 4.3 (Virtual measurement operators)**.** Let $B \subseteq V \cap \Omega$ be a non-empty closed ball with $q_1 \gneqq q_2$ on B as in (4.1). The operators L_B and $L_{\Omega\setminus V}$ are defined by

$$L_B : L^2(B) \to L^2(\Gamma), \qquad f \mapsto v_B|_\Gamma, \tag{4.4}$$

$$L_{\Omega\setminus V} : L^2(\Omega \setminus V) \to L^2(\Gamma), \qquad h \mapsto v_{\Omega\setminus V}|_\Gamma, \tag{4.5}$$

where $v_B, v_{\Omega\setminus V} \in H^1(\Omega)$ are the unique solutions of

$$-\Delta v_B + q_1 v_B = |q_1 - q_2|^{1/2} f \chi_B \text{ in } \Omega \quad \text{with} \quad \partial_\nu v_B|_{\partial\Omega} = 0, \tag{4.6}$$

$$-\Delta v_{\Omega\setminus V} + q_1 v_{\Omega\setminus V} = |q_1 - q_2|^{1/2} h \chi_{\Omega\setminus V} \text{ in } \Omega \quad \text{with} \quad \partial_\nu v_{\Omega\setminus V}|_{\partial\Omega} = 0, \tag{4.7}$$

or equivalently

$$\int_\Omega \nabla v_B \cdot \nabla w + q_1 v_B w \,\mathrm{d}x = \int_B |q_1 - q_2|^{1/2} w f \,\mathrm{d}x \quad \forall w \in H^1(\Omega), \tag{4.8}$$

$$\int_\Omega \nabla v_{\Omega\setminus V} \cdot \nabla w + q_1 v_{\Omega\setminus V} w \,\mathrm{d}x = \int_{\Omega\setminus V} |q_1 - q_2|^{1/2} w h \,\mathrm{d}x \quad \forall w \in H^1(\Omega). \tag{4.9}$$

Lemma 4.4. (a) *The adjoint operators*

$$L_B^* : L^2(\Gamma) \to L^2(B) \quad \textit{and} \quad L_{\Omega\setminus V}^* : L^2(\Gamma) \to L^2(\Omega \setminus V) \tag{4.10}$$

fulfill

$$L_B^* g = \left(|q_1 - q_2|^{1/2} u\right)\Big|_B \quad \textit{and} \quad L_{\Omega\setminus V}^* g = \left(|q_1 - q_2|^{1/2} u\right)\Big|_{\Omega\setminus V}, \tag{4.11}$$

where $u := u_{q_1}^{(g)} \in H^1(\Omega)$ *is the corresponding solution of* (1.3).
(b) *The adjoint operator* L_B^* *is injective and* $\overline{\mathcal{R}(L_B)} = L^2(\Gamma)$.
(c) $\mathcal{R}(L_B) \cap \mathcal{R}(L_{\Omega\setminus V}) = \{0\}$.
(d) $\mathcal{R}(L_B) \not\subseteq \mathcal{R}(L_{\Omega\setminus V})$.
(e) $\nexists C > 0 : \|L_B^* g\| \leq C \|L_{\Omega\setminus V}^* g\| \quad \forall g \in L^2(\Gamma)$.

Proof. (a) For $f \in L^2(B)$, let $v_B^{(f)} \in H^1(\Omega)$ be the solution of

$$\int_\Omega \nabla v_B^{(f)} \cdot \nabla w + q_1 v_B^{(f)} w \,\mathrm{d}x = \int_B |q_1 - q_2|^{1/2} w f \,\mathrm{d}x \quad \forall w \in H^1(\Omega).$$

Then, in particular,

$$L_B f = v_B^{(f)}|_\Gamma.$$

Furthermore, for $g \in L^2(\Gamma)$, let $u_{q_1}^{(g)}$ be the corresponding solution of (1.3). Then, $u_{q_1}^{(g)}$ also solves the equivalent variational formulation

$$\int_\Omega \nabla w \cdot \nabla u_{q_1}^{(g)} + q_1 w u_{q_1}^{(g)} \, \mathrm{d}x = \int_\Gamma g w|_\Gamma \, \mathrm{d}s \quad \forall w \in H^1(\Omega).$$

Hence, for arbitrary $f \in L^2(B)$ and $g \in L^2(\Gamma)$,

$$\begin{aligned}(f, L_B^* g)_{L^2(B)} = (L_B f, g)_{L^2(\Gamma)} &= \int_\Gamma g v_B^{(f)}|_\Gamma \, \mathrm{d}s \\ &= \int_\Omega \nabla v_B^{(f)} \cdot \nabla u_{q_1}^{(g)} + q_1 v_B^{(f)} u_{q_1}^{(g)} \, \mathrm{d}x \\ &= \int_B |q_1 - q_2|^{1/2} u_{q_1}^{(g)} f \, \mathrm{d}x \\ &= \left(f, \left(|q_1 - q_2|^{1/2} u_{q_1}^{(g)}\right)|_B\right)_{L^2(B)}.\end{aligned}$$

This yields $L_B^* g = \left(|q_1 - q_2|^{1/2} u_{q_1}^{(g)}\right)\Big|_B$.

Analogously, it follows $L_{\Omega\setminus V}^* g = \left(|q_1 - q_2|^{1/2} u_{q_1}^{(g)}\right)\Big|_{\Omega\setminus V}$.

(b) First, we prove the injectivity of L_B^*. Let $g \in L^2(\Gamma)$ with $L_B^* g = 0$ and $u := u_{q_1}^{(g)} \in H^1(\Omega)$ be the corresponding solution of (1.3). ¿From (a) it follows $L_B^* g = \left(|q_1 - q_2|^{1/2} u\right)|_B$. Since $q_1 - q_2 \gneq 0$ on B, there exists a measurable set $E \subseteq B$ of positive measure where $|q_1 - q_2|^{1/2} \neq 0$. Hence, $\left(|q_1 - q_2|^{1/2} u\right)|_E \equiv 0$ implies $u|_E \equiv 0$. ¿From Theorem 4.2 it follows that $u \equiv 0$ on Ω and thus $g = \partial_\nu u|_\Gamma = 0$. This shows the injectivity of L_B^* and thus $\overline{\mathcal{R}(L_B)} = \mathcal{N}(L_B^*)^\perp = L^2(\Gamma)$.

(c) Recall that B and $\Omega \setminus V$ are closed in Ω and that $V \setminus B$ is connected. Let

$$\phi = L_B f = L_{\Omega\setminus V} h \in \mathcal{R}(L_B) \cap \mathcal{R}(L_{\Omega\setminus V})$$

and $v_B, v_{\Omega\setminus V} \in H^1(\Omega)$ be the corresponding solutions of Definition 4.3. First, we show that

$$v_B = v_{\Omega\setminus V} \quad \text{on} \quad \Omega \setminus \left(\overline{B} \cup \overline{\Omega \setminus V}\right) = (\Omega \cap V) \setminus B : \tag{4.12}$$

On $\Omega \cup V$, we define the continuations

$$q := \begin{cases} q_1, & \text{on } \Omega, \\ 1, & \text{on } V \setminus \Omega, \end{cases}$$

$$\tilde{v} := \begin{cases} v, & \text{on } \Omega, \\ 0, & \text{on } V \setminus \Omega, \end{cases} \quad \text{and} \quad \tilde{v}_j := \begin{cases} \partial_{x_j} v, & \text{on } \Omega, \\ 0, & \text{on } V \setminus \Omega, \end{cases}$$

where $v := v_B - v_{\Omega\setminus V}$.

Obviously, $\tilde{v}, \tilde{v}_j \in L^2(\Omega \cup V)$. To verify that $\tilde{v} \in H^1(\Omega \cup V)$, it is left to show $\partial_{x_j} \tilde{v} = \tilde{v}_j$. This can be shown by using

$$v|_\Gamma = v_B|_\Gamma - v_{\Omega\setminus V}|_\Gamma = \phi - \phi = 0.$$

Let $\varphi \in \mathcal{D}(\Omega \cup V)$, then,

$$\begin{aligned}\int_{\Omega\cup V} \tilde{v}\partial_{x_j}\varphi \,\mathrm{d}x &= \int_{\Omega} v\partial_{x_j}\varphi \,\mathrm{d}x \\ &= -\int_{\Omega} \varphi\partial_{x_j} v \,\mathrm{d}x + \int_{\partial\Omega} (\varphi v)|_{\Gamma}\nu_j \,\mathrm{d}s \\ &= -\int_{\Omega} v_j\varphi \,\mathrm{d}x = -\int_{\Omega\cup V} \tilde{v}_j\varphi \,\mathrm{d}x.\end{aligned}$$

Now, we go on showing (4.12). Since $v = \tilde{v}|_{\Omega}$ fulfills

$$\begin{aligned}&\int_{\Omega} \nabla v \cdot \nabla w + q_1 v w \,\mathrm{d}x \\ &\qquad = \int_{\Omega} |q_1 - q_2|^{1/2} w(f\chi_B - h\chi_{\Omega\setminus V}) \,\mathrm{d}x \quad \forall w \in H^1(\Omega),\end{aligned}$$

it holds

$$\int_{V\setminus B} \nabla\tilde{v} \cdot \nabla\varphi + q\tilde{v}\varphi \,\mathrm{d}x = 0 \quad \forall \varphi \in \mathcal{D}\left(V \setminus B\right).$$

We obtain that $\tilde{v}$ (as a function in $H^1\left(V \setminus B\right)$) solves

$$-\Delta\tilde{v} + q\tilde{v} = 0 \quad \text{on} \quad V \setminus B$$

and vanishes on $V \setminus \Omega$. Since $V \setminus \Omega$ is a non-empty open set (V is open and has a non-empty intersection Γ with the Lipschitz-domain Ω) and $V \setminus B$ is connected, Theorem 4.2 shows that $\tilde{v} \equiv 0$ on $V \setminus B$ and thus

$$v_B = v_{\Omega\setminus V} \quad \text{on} \quad (V \cap \Omega) \setminus B.$$

To finally show $\phi = 0$, we define

$$u := \begin{cases} v_B & \text{on } \Omega \setminus B, \\ v_{\Omega\setminus V} & \text{on } B. \end{cases}$$

We can partition test functions (in $\mathcal{D}(\Omega)$ and $H^1(\Omega)$), by using smooth partitions of unity, to prove that u is an $H^1(\Omega)$-function and the unique solution of

$$\begin{aligned}-\Delta u + q_1 u = 0 \quad \text{on} \quad \Omega, \\ \partial_\nu u|_{\partial\Omega} = 0.\end{aligned}$$

Hence, u has to be equal to the trivial solution and thus

$$\phi = v_B|_{\Gamma} = u|_{\Gamma} \equiv 0.$$

(d) This simply follows from (b) and (c).

(e) Let us assume there exists a constant $C > 0$ such that

$$\|L_B^* g\| \le C\|L_{\Omega\setminus V}^* g\| \quad \forall g \in L^2(\Gamma).$$

Then,

$$\mathcal{R}(L_B) \subseteq \mathcal{R}(L_{\Omega\setminus V})$$

immediately follows from Theorem 4.1 and this is a contradiction to (d).

□

Proof of Lemma 2.2. The assertion follows from Lemma 4.4:

The trivial case is that when $L^*_{\Omega\setminus V}$ is not injective. Then, there exists an element $g \in L^2(\Gamma) \setminus \{0\}$ with $\|L^*_{\Omega\setminus V} g\| = 0$. By the injectivity of L^*_B we have $\|L^*_B g\| =: c_g \geq 0$. In this case, we can set $g_m := mg$ for all $m \in \mathbb{N}$.

When $L^*_{\Omega\setminus V}$ is injective, we derive a suitable sequence $(g_m)_{m\in\mathbb{N}} \subseteq L^2(\Gamma)$ as follows.

Let $C_m = m^2$ for $m \in \mathbb{N}$. Lemma 4.4 (e) implies the existence of a sequence $(g'_m)_{m\in\mathbb{N}} \subseteq L^2(\Gamma)$ with

$$\|L^*_B g'_m\| > C_m \|L^*_{\Omega\setminus V} g'_m\| \quad \forall m \in \mathbb{N}. \tag{4.13}$$

In particular, this implies $g'_m \neq 0$ for all $m \in \mathbb{N}$. Since $L^*_{\Omega\setminus V}$ is injective, we can set $g_m := \frac{g'_m}{m\|L^*_{\Omega\setminus V} g'_m\|}$. By multiplying (4.13) with $\frac{1}{m\|L^*_{\Omega\setminus V} g'_m\|}$, we obtain

$$\|L^*_B g_m\| > m \quad \forall m \in \mathbb{N}.$$

Furthermore, it holds

$$\|L^*_{\Omega\setminus V} g_m\| = \frac{1}{m} \quad \forall m \in \mathbb{N}.$$

For both cases, we obtain a sequence $(g_m)_{m\in\mathbb{N}}$ such that

$$\lim_{m\to\infty} \int_{V\cap\Omega} (q_1 - q_2) {u_m}^2 \,\mathrm{d}x = \lim_{m\to\infty} \|L^*_B g_m\|^2 = \infty,$$
$$\lim_{m\to\infty} \int_{\Omega\setminus V} (q_1 - q_2) {u_m}^2 \,\mathrm{d}x = \lim_{m\to\infty} \|L^*_{\Omega\setminus V} g_m\|^2 = 0$$

where $u_m := u_{q_1}^{(g_m)} \in H^1(\Omega)$ is the corresponding solution of (1.3). □

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University of Stuttgart, Department of Mathematics - IMNG, Chair of Optimization and Inverse Problems, Allmandring 5b, 70569 Stuttgart, Germany

E-mail address: harrach@math.uni-stuttgart.de

University of Stuttgart, Department of Mathematics - IMNG, Chair of Optimization and Inverse Problems, Allmandring 5b, 70569 Stuttgart, Germany

E-mail address: marcel.ullrich@mathematik.uni-stuttgart.de