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EXACT SHAPE-RECONSTRUCTION BY ONE-STEP LINEARIZATION IN ELECTRICAL IMPEDANCE TOMOGRAPHY*

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Abstract. For electrical impedance tomography (EIT), the linearized reconstruction method using the Fréchet derivative of the Neumann-to-Dirichlet map with respect to the conductivity has been widely used in the last three decades. However, few rigorous mathematical results are known regarding the errors caused by the linear approximation. In this work we prove that linearizing the inverse problem of EIT does not lead to shape errors for piecewise-analytic conductivities. If a solution of the linearized equations exists, then it has the same outer support as the true conductivity change, no matter how large the latter is. Under an additional definiteness condition we also show how to approximately solve the linearized equation so that the outer support converges toward the right one. Our convergence result is global and also applies for approximations by noisy finite-dimensional data. Furthermore, we obtain bounds on how well the linear reconstructions and the true conductivity difference agree on the boundary of the outer support.

Key words. inverse problems, electrical impedance tomography, linearization, shape reconstruction

AMS subject classifications. 35R30, 35Q60, 35J25, 35R05

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1. Introduction.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$ and outer normal vector ν. Let σ_0 be a reference conductivity distribution, and let σ_1 be a perturbation of it. Assume that σ_1 and σ_0 are piecewise analytic. The inverse problem corresponding to electrical impedance tomography (EIT) is to recover the change $\sigma_1 - \sigma_0$ from the difference of two Neumann-to-Dirichlet (NtD) operators, $\Lambda(\sigma_1) - \Lambda(\sigma_0)$. Here the NtD operator is the map $\Lambda(\sigma) : \ g \mapsto u_\sigma|_{\partial\Omega}$, where

$$\nabla \cdot \sigma \nabla u_\sigma = 0 \quad \text{in } \Omega, \qquad \sigma \partial_\nu u_\sigma|_{\partial\Omega} = g \quad \text{on } \partial\Omega.$$

According to linear EIT reconstruction methods, we can directly recover κ such that

$$\Lambda'(\sigma_0)\kappa \approx \Lambda(\sigma_1) - \Lambda(\sigma_0), \tag{1.1}$$

where $\Lambda'(\sigma_0)$ denotes the Fréchet derivative of $\Lambda(\sigma)$ evaluated at σ_0. One of the major unsolved challenges in linear EIT reconstruction is to analyze the relation between the linear EIT solution κ and the true change $\sigma_1 - \sigma_0$.

In this paper, we prove that the linear EIT reconstruction method provides the correct (outer) support of $\sigma_1 - \sigma_0$, no matter how large $\sigma_1 - \sigma_0$ is. The term *outer*

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support will be defined later on; it coincides with the support if the latter is compactly contained in Ω and has a connected complement.

To be more precise, our result is twofold. We first prove that every *exact* (piecewise-analytic) solution κ of (1.1) has the correct outer support. Since it is not clear under which conditions such an exact solutions exists, we then derive (under an additional definiteness assumption) a globally convergent method to calculate the outer support by approximately solving (1.1). Our second result also applies for the realistic case that the quantities in (1.1) are approximated by noisy finite-dimensional data. Furthermore, we obtain bounds on how well $\sigma_1 - \sigma_0$ and κ agree on the boundary of the outer support.

Let us describe the origins and applications of this result. The goal of EIT is to produce spatial and temporal images of the conductivity σ within an electrically conducting subject such as the human body, using the boundary voltage data resulting from the injection of electrical currents. EIT imaging has shown its potential for medical applications such as the noninvasive monitoring of physiological events, lung and heart function, and blood flow; cf. [6, 52, 13, 32].

Great theoretical progress has been made on the injectivity of the nonlinear, ill-posed forward mapping $\Lambda: \ \sigma \mapsto \Lambda(\sigma)$, which is known as the Calderón problem [11, 12]. For full boundary data, see [45, 59, 54, 5] and the overview [60]. For partial boundary data we refer to [10, 44, 41, 38, 21] and the preprint [34]. In practice, however, due to measurement errors and forward modeling errors such as boundary geometry errors, stable reconstruction of the absolute conductivity distribution (static EIT) still seems to be out of reach.

Despite the problems in static EIT, the imaging of temporal changes in the conductivity by time-difference EIT based on a linear approximation has been very successful, and there have been numerous results in theory and algorithms in the last three decades; cf., e.g., [6, 7, 62, 61, 24, 56, 37, 14, 25, 57, 36, 19, 15, 35, 51].

The most widely used reconstruction algorithm for EIT would be the one-step linearization described by (1.1). This is also known as the one-step linear Gauss–Newton method or (with a specific implementation) as Newton's one-step error reconstructor, NOSER (cf. Simske [58] and Cheney et al. [14]). The references above contain several ways of approximating and regularizing the solution of (1.1), and recently Adler et al. [1] addressed the purpose of reaching consensus in the EIT community on the linear reconstruction algorithms by controlling the approximation and regularization on the basis of a set of training data. However, the crucial question remains of how close the reconstructed κ is to the (possibly large) true change $\sigma_1 - \sigma_0$. To the authors' knowledge, there has not been any rigorous global result for this issue until now.

The outer support of the conductivity change, that is herein shown to be unaffected by linearization errors, is of enormous physiological relevance for medical applications. In fact, instead of aiming at (e.g., cross-sectional) conductivity imaging, several recent studies in EIT focus on this anomaly- or shape-detection problem; cf., e.g., [31, 39, 48, 49, 4, 18, 3, 33, 17, 27], the references therein, and the works on the factorization method [9, 8, 42, 26, 20, 28, 55, 22, 43], from which many ideas of the present work stem.

Our result extends the practically well-established linear reconstruction method by the rigorous mathematical property of exact shape reconstruction. Also note that our convergence result in Theorem 4.4 and Remark 4.5 is comparatively strong, as it holds under the realistic assumption that the data is approximated by noisy finite-dimensional measurements.

The outline of this work is as follows. In section 2 we introduce some basic notation and recall an important monotony result. Applying the localized potentials from [21], we then prove in section 3 that the outer support of a piecewise-analytic function and its boundary values can be identified from bounds on its expansion with respect to squared (gradients of) EIT solutions. This yields that every exact solution of (1.1) possesses the correct outer support. Finally, in section 4 we show how to approximately solve (1.1) and prove convergence.

2. Basic notation and monotony. We begin by introducing our notation and summarizing some known monotony results. As above, let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain describing the investigated body, and let $\sigma \in L^\infty_+(\Omega)$ be its conductivity distribution. Ω is assumed to have smooth boundary $\partial\Omega$ and outer normal vector ν. L^∞_+ denotes the subspace of L^∞-functions with positive essential infima.

The nonlinear forward operator of EIT is defined by

$$\Lambda : \; L^\infty_+(\Omega) \to \mathcal{L}(L^2_\diamond(\partial\Omega)), \quad \sigma \mapsto \Lambda(\sigma),$$

where $\Lambda(\sigma) : \; g \mapsto u_\sigma|_{\partial\Omega}$ and $u_\sigma \in H^1_\diamond(\Omega)$ solves

$$\nabla \cdot \sigma \nabla u_\sigma = 0, \qquad \sigma \partial_\nu u_\sigma|_{\partial\Omega} = g. \tag{2.1}$$

$H^1_\diamond(\Omega)$ and $L^2_\diamond(\partial\Omega)$ denote the spaces of H^1- and L^2-functions with vanishing integral mean on $\partial\Omega$.

The quadratic form associated with $\Lambda(\sigma)$ is

$$\langle \Lambda(\sigma) g, g \rangle = \int_{\partial\Omega} g \Lambda(\sigma) g \, \mathrm{d}s = \int_\Omega \sigma |\nabla u_\sigma|^2 \, \mathrm{d}x,$$

where, here and in the following, u_σ is the (g-dependent) solution of (2.1).

It is well known that Λ is Fréchet-differentiable; cf., e.g., Lechleiter and Rieder [50] for a recent proof that uses only the abstract variational formulation. Given some direction $\kappa \in L^\infty(\Omega)$, the derivative $\Lambda'(\sigma)\kappa \in \mathcal{L}(L^2_\diamond(\partial\Omega))$ is associated to the quadratic form

$$\langle (\Lambda'(\sigma)\kappa) \, g, g \rangle = - \int_\Omega \kappa |\nabla u_\sigma|^2 \, \mathrm{d}x,$$

where $\langle \cdot, \cdot \rangle$ denotes the $L^2_\diamond(\partial\Omega)$-inner product. If the direction κ is compactly supported in Ω, then we can also interpret this as

$$\Lambda'(\sigma)\kappa : \; g \mapsto v|_{\partial\Omega},$$

where v solves

$$\nabla \cdot \sigma \nabla v = - \nabla \kappa \nabla u_\sigma, \qquad \sigma \partial_\nu v|_{\partial\Omega} = 0. \tag{2.2}$$

Given conductivities $\sigma_0, \sigma_1 \in L^\infty_+(\Omega)$, we will write u_0 (resp., u_1) instead of u_{σ_0} (resp., u_{σ_1}) for the sake of readability. The quadratic forms for Λ and its Fréchet derivative are related by the following monotony relations.

LEMMA 2.1. *For $\sigma_0, \sigma_1 \in L^\infty_+(\Omega)$ we have in the sense of quadratic forms*

$$\Lambda'(\sigma_0)(\sigma_1 - \sigma_0) \;\leq\; \Lambda(\sigma_1) - \Lambda(\sigma_0) \;\leq\; \Lambda'(\sigma_0) \left(\frac{\sigma_0}{\sigma_1} (\sigma_1 - \sigma_0) \right);$$

i.e., for all $g \in L^2_\diamond(\partial\Omega)$,

$$\langle \Lambda'(\sigma_0)(\sigma_1-\sigma_0)g,g\rangle \;\le\; \langle (\Lambda(\sigma_1)-\Lambda(\sigma_0))\,g,g\rangle \;\le\; \left\langle \Lambda'(\sigma_0)\left(\frac{\sigma_0}{\sigma_1}(\sigma_1-\sigma_0)\right)g,g\right\rangle.$$

Proof. For Dirichlet-to-Neumann operators a similar monotony result is proven in Ide et al. [33, Lemma 2.6]; cf. also the similar arguments in Kang, Seo, and Sheen [40], Kirsch [42], and the current authors [30].

From

$$\int_\Omega \sigma_1 \nabla u_1 \cdot \nabla u_0 \,\mathrm{d}x = \langle \Lambda(\sigma_0)g,g\rangle = \int_\Omega \sigma_0 \nabla u_0\cdot\nabla u_0\,\mathrm{d}x,$$

we have

$$\begin{aligned}
&\int_\Omega \sigma_1 |\nabla(u_1-u_0)|^2\,\mathrm{d}x\\
&= \int_\Omega \sigma_1|\nabla u_1|^2\,\mathrm{d}x - \int_\Omega \sigma_0|\nabla u_0|^2\,\mathrm{d}x + \int_\Omega (\sigma_1-\sigma_0)|\nabla u_0|^2\,\mathrm{d}x\\
&= \langle(\Lambda(\sigma_1)-\Lambda(\sigma_0))g,g\rangle - \langle \Lambda'(\sigma_0)(\sigma_1-\sigma_0)g,g\rangle\,.
\end{aligned}$$

Since the above equalities hold for all $g \in L^2_\diamond(\partial\Omega)$, we obtain the first inequality. Similarly, the second inequality can be obtained by interchanging σ_1 and σ_0:

$$\begin{aligned}
\langle(\Lambda(\sigma_0)-\Lambda(\sigma_1))g,g\rangle &= \int_\Omega (\sigma_1-\sigma_0)|\nabla u_1|^2\,\mathrm{d}x + \int_\Omega \sigma_0|\nabla(u_0-u_1)|^2\,\mathrm{d}x\\
&= \int_\Omega \left(\sigma_1\left|\nabla u_1 - \frac{\sigma_0}{\sigma_1}\nabla u_0\right|^2 + \left(\sigma_0 - \frac{\sigma_0^2}{\sigma_1}\right)|\nabla u_0|^2\right)\,\mathrm{d}x\\
&\ge \left\langle \Lambda'(\sigma_0)\left(\frac{\sigma_0}{\sigma_1}(\sigma_0-\sigma_1)\right)g,g\right\rangle. \qquad \square
\end{aligned}$$

The linearized reconstruction method for EIT is based on the approximation

$$\Lambda'(\sigma_0)(\sigma_1-\sigma_0) \approx \Lambda(\sigma_1)-\Lambda(\sigma_0).$$

Here σ_0 is assumed to be known (e.g., homogeneous background conductivity), and we seek to recover the conductivity change $\sigma_1-\sigma_0$ from the difference data $\Lambda(\sigma_1)-\Lambda(\sigma_0)$. According to Lemma 2.1, if there exists $\kappa \in L^\infty(\Omega)$ such that

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma_1)-\Lambda(\sigma_0), \tag{2.3}$$

the following inequalities hold for all solutions u_0 of (2.1) with $\sigma=\sigma_0$:

$$\int_\Omega (\sigma_1-\sigma_0)|\nabla u_0|^2\,\mathrm{d}x \ge \int_\Omega \kappa|\nabla u_0|^2\,\mathrm{d}x \ge \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_1-\sigma_0)|\nabla u_0|^2\,\mathrm{d}x. \tag{2.4}$$

In section 3 we will prove that (2.4) implies that κ and $\sigma_1-\sigma_0$ have the same outer support and that κ lies between $\sigma_1-\sigma_0$ and $\frac{\sigma_0}{\sigma_1}(\sigma_1-\sigma_0)$ on the boundary of the outer support. Under the additional definiteness assumption $\sigma_1-\sigma_0\ge 0$, we will then show in section 4 how to approximately solve (2.3) in a way that it retains the property (2.4). In effect, we obtain a one-step linearization algorithm that yields the correct outer support.

3. Identification of the outer support.

3.1. The identification theorem. The main result of this section is that the outer support of a piecewise-analytic function and its boundary values can be identified from bounds on its expansion with respect to squared (gradients of) EIT solutions.

We first make the terms *outer support* and *piecewise-analytic* precise. The concept of the *outer support* (or $\partial\Omega$-support) was introduced by Harrach and Hyvönen [23] as an analogue of the infinity support of Kusiak and Sylvester [47].

DEFINITION 3.1. *The* outer support *(or $\partial\Omega$-support) of $\kappa \in L^\infty(\Omega)$ is the complement of the set of all $x \in \Omega$ for which there exists a (relatively) open connected set $U \subset \overline{\Omega}$ with $x \in U$, $\partial\Omega \cap U \neq \emptyset$, and $\kappa|_U$ vanishing almost everywhere. We denote this set by* $\operatorname{supp}_{\partial\Omega} \kappa$.

The term *piecewise-analytic* is understood as in [29] and the following.

DEFINITION 3.2.

(a) *An open set O is said to have a* piecewise-smooth boundary *if ∂O is a countable union of C^∞-curves and O lies locally on one side of ∂O.*

(b) *A function $\kappa \in L^\infty(\Omega)$ is called* piecewise-analytic *if there exist finitely many pairwise disjoint subdomains $O_1, \ldots, O_M \subset \Omega$ with piecewise-smooth boundaries, such that $\overline{\Omega} = \overline{O_1 \cup \cdots \cup O_M}$, and $\kappa|_{O_m}$ has an extension which is (real-)analytic in a neighborhood of $\overline{O_m}$, $m = 1, \ldots, M$.*

To formulate our identification theorem in the most general sense, we also need the concept of unique continuation, which we define as in [21].

DEFINITION 3.3. *A conductivity $\sigma \in L^\infty_+(\Omega)$ is said to satisfy the* unique continuation property (UCP) *if only constant solutions of $\nabla \cdot (\sigma \nabla u_\sigma) = 0$ can be constant on an open subset of Ω and if only the trivial solution possesses zero Cauchy data $u_\sigma|_{\partial\Omega} = 0$ and $\sigma \partial_\nu u_\sigma|_{\partial\Omega} = 0$.*

Note that in two dimensions every $\sigma \in L^\infty_+(\Omega)$ satisfies the UCP; cf. Alessandrini and Magnanini [2]. In three or higher dimensions this is fulfilled for Lipschitz continuous functions (cf., e.g., Miranda [53]) and obviously extends to piecewise Lipschitz continuous conductivities (cf., e.g., Druskin [16]). In particular it holds for piecewise-analytic functions.

Now we can state our identification theorem.

THEOREM 3.4. *Let $\sigma \in L^\infty_+(\Omega)$ satisfy the UCP, and let $\kappa, \lambda \in L^\infty(\Omega)$ be two piecewise-analytic functions. If there exist $a, b, c, d \in L^\infty_+(\Omega)$ such that*

$$\int_\Omega a\kappa |\nabla u_\sigma|^2 \, \mathrm{d}x \leq \int_\Omega c\lambda |\nabla u_\sigma|^2 \, \mathrm{d}x, \tag{3.1}$$

$$\int_\Omega b\kappa |\nabla u_\sigma|^2 \, \mathrm{d}x \geq \int_\Omega d\lambda |\nabla u_\sigma|^2 \, \mathrm{d}x \tag{3.2}$$

holds for all solutions $u_\sigma \in H^1_\diamond(\Omega)$ of $\nabla \cdot \sigma \nabla u_\sigma = 0$, then

$$\operatorname{supp}_{\partial\Omega} \kappa = \operatorname{supp}_{\partial\Omega} \lambda.$$

Furthermore, for all $x \in \partial D$, $D := \operatorname{supp}_{\partial\Omega} \kappa$ with a neighborhood U on which $\frac{d}{b}\lambda|_{D\cap U}$, $\kappa|_{D\cap U}$, and $\frac{c}{a}\lambda|_{D\cap U}$ are continuous,

$$\frac{d(x)}{b(x)}\lambda(x) \leq \kappa(x) \leq \frac{c(x)}{a(x)}\lambda(x), \tag{3.3}$$

where the evaluations are taken from $D \cap U$.

In the next subsection we prove the theorem using the localized potentials from [21]. Before we do that, let us formulate three corollaries of Lemma 2.1 and Theorem 3.4 and give some comments.

COROLLARY 3.5. *Let $\sigma_0, \sigma_1 \in L^\infty_+(\Omega)$ satisfy the UCP. Then we have the following:*

(a) *The Calderón problem is uniquely solvable for piecewise-analytic conductivities; i.e., if $\sigma_1 - \sigma_0$ is piecewise-analytic and $\Lambda(\sigma_1) = \Lambda(\sigma_0)$, then $\sigma_1 = \sigma_0$.*

(b) *The linearized Calderón problem is uniquely solvable for piecewise-analytic conductivities; i.e., $\Lambda'(\sigma_0)$ is injective in the space of piecewise-analytic conductivities.*

(c) *A single* exact *linearization step for the EIT problem already correctly reconstructs the outer boundary. More precisely, let $\sigma_0 - \sigma_1$ be piecewise-analytic. If there exists a piecewise-analytic solution of*

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma_1) - \Lambda(\sigma_0), \tag{3.4}$$

then $\operatorname{supp}_{\partial\Omega} \kappa = \operatorname{supp}_{\partial\Omega}(\sigma_1 - \sigma_0)$ *and on* $\operatorname{supp}_{\partial\Omega} \kappa$

$$\frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0) \leq \kappa \leq \sigma_1 - \sigma_0$$

holds for all boundary points as described in Theorem 3.4.

Note that there is not much new in (a) and (b); the authors merely find it interesting to note that they directly follow from Theorem 3.4. The first assertion is essentially the famous result of Kohn and Vogelius [46], and [21] already notes that this can be proven by localized potentials. The second one has been derived for piecewise polynomials by Lechleiter and Rieder [50] (also using localized potentials).

It is Corollary 3.5(c) that contains the key idea of this work: If a single linearization step could be carried out exactly, then the outer boundary of the conductivity difference would be reconstructed exactly. The linearization error does not affect the shape reconstruction regardless of the magnitude of the difference $\sigma_1 - \sigma_0$.

There are, however, two fundamental problems with this argument. The first one is of a theoretical nature. Even with exact data, it is not clear whether $\Lambda(\sigma_0) - \Lambda(\sigma_1)$ belongs to the range of $\Lambda'(\sigma_0)$, i.e., whether (3.4) has an (L^∞ or piecewise-analytic) solution. To the authors' knowledge, it is not even clear whether $\Lambda'(\sigma_0)$ has dense range. The second problem is that even if an exact piecewise-analytic solution of (3.4) exists, practically measurable data is of finite accuracy and limited by the number of electrodes, so that one is confronted with noisy finite-dimensional approximations to $\Lambda(\sigma_0)$, $\Lambda(\sigma_1)$, and $\Lambda'(\sigma_0)$.

Under the additional definiteness assumption that $\sigma_1 - \sigma_0 \geq 0$, we will deal with both problems in section 4 and derive a one-step linearization algorithm that converges toward the exact shape.

3.2. Proof of the identification theorem. Throughout this subsection let $\sigma \in L^\infty_+(\Omega)$ satisfy the UCP. Our main tool is the localized potentials developed by one of the authors in [21]. We will apply the following version of [21, Theorem 2.7].

THEOREM 3.6. *Let $O \subseteq \overline{\Omega}$ be a (relatively) open subdomain with $O \cap \partial\Omega \neq \emptyset$. Furthermore, let B be an open subdomain with $\overline{B} \subset O$.*

Then there exists a sequence of solutions $(u_\sigma^{(j)})_{j\in\mathbb{N}} \subset H^1_\diamond(\Omega)$ of $\nabla \cdot \sigma \nabla u_\sigma^{(j)} = 0$ such that

$$\int_B |\nabla u_\sigma^{(j)}|^2 \,\mathrm{d}x \to \infty, \quad \textit{but} \quad \int_{\Omega \setminus \overline{O}} |\nabla u_\sigma^{(j)}|^2 \,\mathrm{d}x \to 0.$$

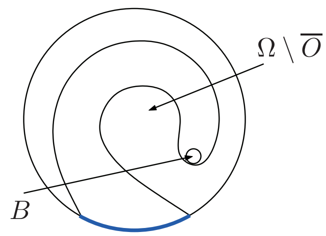

FIG. 3.1. *Sketch of the domains in Theorem* 3.6.

See Figure 3.1 *for a sketch of this situation.*

Proof. Note that $O \cap \partial\Omega \neq \emptyset$ implies that ∂O contains an open subset of the boundary $\partial\Omega$. Hence, the assertion immediately follows from [21, Theorem 2.7]. $\square$

LEMMA 3.7. *Let κ be a piecewise-analytic function on Ω, and let $U \subseteq \overline{\Omega}$ be a (relatively) open subdomain with $U \cap \partial\Omega \neq \emptyset$.*

If there exist $C > 0$ and $a, b \in L^\infty_+(\Omega)$ such that for all solutions $u_\sigma \in H^1_\diamond(\Omega)$ of $\nabla \cdot \sigma \nabla u_\sigma = 0$

$$\int_\Omega a\kappa |\nabla u_\sigma|^2 \, \mathrm{d}x \leq C \int_{\Omega\setminus\overline{U}} |\nabla u_\sigma|^2 \, \mathrm{d}x, \tag{3.5}$$

$$\int_\Omega b\kappa |\nabla u_\sigma|^2 \, \mathrm{d}x \geq -C \int_{\Omega\setminus\overline{U}} |\nabla u_\sigma|^2 \, \mathrm{d}x, \tag{3.6}$$

then $\kappa|_U = 0$.

Proof. We proceed similarly to the proof of [29, Theorem 4.2]. First note that $\partial U \cap \partial\Omega$ must contain a joint open piece S.

Let $O_1, \ldots, O_M$ be the partition on which κ is analytic. We will consider the open sets $O'_m := O_m \cap U$. One of the boundaries of these sets must contain an open subpiece of S. Without loss of generality let this set be O'_1 and the subpiece be S.

Assume that $\kappa \neq 0$ on O'_1. Then, by analyticity, there must be a nonvanishing normal derivative $\partial^l_\nu \kappa|_S$, $l \in \mathbb{N}_0$. Hence there exists a neighborhood of a smooth open subpiece of S in which either $\kappa \geq 0$ or $\kappa \leq 0$. We first assume $\kappa \geq 0$, and we shrink S and O'_1 to the aforementioned subpiece and its neighborhood. Again by analyticity, there must be a subdomain $B \subseteq O'_1$ on which $\kappa \in L^\infty_+(B)$. Applying the localized potentials from Theorem 3.6 with B and O'_1, we obtain a sequence for which

$$\int_{O'_1} a\kappa |\nabla u^{(j)}_\sigma|^2 \, \mathrm{d}x \geq \int_B a\kappa |\nabla u^{(j)}_\sigma|^2 \, \mathrm{d}x \to \infty, \quad \text{but} \quad \int_{\Omega\setminus\overline{O'_1}} |\nabla u^{(j)}_\sigma|^2 \, \mathrm{d}x \to 0,$$

which contradicts the assumption (3.5). In the case when $\kappa \leq 0$ we analogously obtain a contradiction from using a sequence with

$$\int_{O'_1} b\kappa |\nabla u^{(j)}_\sigma|^2 \, \mathrm{d}x \to -\infty, \quad \text{but} \quad \int_{\Omega\setminus\overline{O'_1}} |\nabla u^{(j)}_\sigma|^2 \, \mathrm{d}x \to 0.$$

Hence, $\kappa = 0$ on O'_1.

Since U is connected, one of the boundaries of the sets $O'_2, \ldots, O'_M$ must contain a smooth open subpiece of $\partial O'_1 \cap U$. Without loss of generality let this be O'_2.

Assume that $\kappa \neq 0$ on O'_2. Then by possibly shrinking O'_2 but keeping the interior of $\overline{O'_2} \cup \overline{O'_1}$ connected, we obtain from the above arguments that (w.l.o.g.) $\kappa \geq 0$ on

O_2' and $\kappa \in L_+^\infty(B)$ on some open $B \subseteq O_2'$. Using the localized potentials from Theorem 3.6 with B and the interior of $\overline{O_2'} \cup \overline{O_1'}$, we obtain the same contradiction as above. Hence, $\kappa = 0$ on O_2'.

Repeating this argument yields that $\kappa = 0$ on all O_m', $m = 1, \ldots, M$, and thus the assertion. ◻

Proof of Theorem 3.4. Let $x \notin \operatorname{supp}_{\partial\Omega} \lambda$. Then there exists a (relatively) open $U \subset \overline{\Omega}$ with $x \in U$, $\partial\Omega \cap U \neq \emptyset$, and $\lambda|_U = 0$.

From assumption (3.1) we obtain

$$\int_\Omega a\kappa |\nabla u_\sigma|^2 \,\mathrm{d}x \leq \int_\Omega c\lambda |\nabla u_\sigma|^2 \,\mathrm{d}x \leq \|c\lambda\|_{L^\infty(\Omega)} \int_{\Omega\setminus\overline{U}} |\nabla u_\sigma|^2 \,\mathrm{d}x,$$

and, analogously, assumption (3.2) yields (3.6). Hence we obtain from Lemma 3.7 that $\kappa|_U = 0$ and thus $x \notin \operatorname{supp}_{\partial\Omega} \kappa$.

Interchanging κ and λ yields that $x \notin \operatorname{supp}_{\partial\Omega} \kappa$ implies $x \notin \operatorname{supp}_{\partial\Omega} \lambda$. Hence, κ and λ have the same outer support.

To show the second part of the assertion, set $D := \operatorname{supp}_{\partial\Omega} \kappa$. Note first that, by analyticity, D must be the closure of the union of some of the O_m, so that in particular D is the closure of its interior $\operatorname{int}(D)$.

Now let $x \in \partial D$ have a neighborhood U on which $\frac{d}{b}\lambda|_{D\cap U}$, $\kappa|_{D\cap U}$, and $\frac{c}{a}\lambda|_{D\cap U}$ are continuous.

Assume that $\frac{d(x)}{b(x)}\lambda(x) > \kappa(x)$. Then there must be a neighborhood of x (w.l.o.g. let this be U) so that $d\lambda - b\kappa \in L_+^\infty(\operatorname{int}(D) \cap U)$. Using the localized potentials from Theorem 3.6 with $B := \operatorname{int}(D) \cap U$ and $O := U \cup (\Omega \setminus \overline{D})$, we obtain a sequence with

$$\int_O (d\lambda - b\kappa)|\nabla u_\sigma^{(j)}|^2 \to \infty \quad \text{and} \quad \int_{\Omega\setminus\overline{O}} (d\lambda - b\kappa)|\nabla u_\sigma^{(j)}|^2 \to 0,$$

which contradicts the assumption (3.2).

With the same arguments, $\kappa(x) > \frac{c(x)}{a(x)}\lambda(x)$ contradicts the assumption (3.1), so the assertion (3.3) follows. ◻

4. Shape reconstruction by one-step linearization. In this section we will require an *additional definiteness assumption*. We will assume that $\sigma_0, \sigma_1 \in L_+^\infty(\Omega)$ are piecewise-analytic and $\sigma_1 \geq \sigma_0$. To exclude the trivial case $\sigma_0 = \sigma_1$ as well as some pathological cases we also require that $\sigma_1 - \sigma_0$ have positive essential infimum on some open subdomain of Ω.

As we saw in section 3, an exact solution of the linearized equation

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma_1) - \Lambda(\sigma_0) \tag{4.1}$$

would possess the right outer boundary, but it is not clear whether such an exact solution exists.

To apply Theorem 3.4, however, it does suffice to find κ such that

$$\Lambda'(\sigma_0)(\sigma_1 - \sigma_0) \leq \Lambda'(\sigma_0)\kappa \leq \Lambda'(\sigma_0)\left(\frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)\right).$$

Such κ do exist. In fact, this is fulfilled by every κ between $\frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)$ and $\sigma_1 - \sigma_0$.

But again it is not clear that we will find such a κ by simply minimizing the linearization residual

$$\Lambda(\sigma_1) - \Lambda(\sigma_0) - \Lambda'(\sigma_0)\kappa \tag{4.2}$$

in the standard $\mathcal{L}(L^2_\diamond(\partial\Omega))$-norm. In the following two subsections we will show that a minimization of (4.2) in a special norm yields a κ with the right outer support.

4.1. Exact infinite-dimensional data. We start by describing the method for exact infinite-dimensional data. The key idea is to minimize (4.2) in a stronger norm. To that end we define

$$\|g\|_\star^2 := \langle(\Lambda(\sigma_0) - \Lambda(\sigma_1))g, g\rangle \geq \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)|\nabla u_0|^2 \, \mathrm{d}x,$$

where the lower bound follows from Lemma 2.1. This obviously gives a norm on $L^2_\diamond(\partial\Omega)$. (Note that the definiteness follows from unique continuation.)

The general idea is to obtain κ by minimizing

$$\sup_{\|g\|_\star=1} |\langle(\Lambda(\sigma_1) - \Lambda(\sigma_0) - \Lambda'(\sigma_0)\kappa)g, g\rangle| \to \min!; \tag{4.3}$$

cf. Remark 4.3 below for a more precise statement.

Note that (4.3) can be interpreted as minimizing (4.2), in the $\mathcal{L}(L^2_\star(\partial\Omega))$-norm, where $L^2_\star(\partial\Omega)$ denotes $L^2_\diamond(\partial\Omega)$ equipped with the $\|\cdot\|_\star$-norm. However, the linearization residual in (4.2) does not have to be a bounded operator on $L^2_\star(\partial\Omega)$. Accordingly, the supremum in (4.3) does not have to be finite for general κ. Therefore, we first show in the next lemma that there exists κ with finite supremum in (4.3).

LEMMA 4.1. *For all $g \in L^2_\diamond(\Omega)$, $\sigma_1 - \sigma_0$ fulfills*

$$0 \leq \langle(\Lambda(\sigma_1) - \Lambda(\sigma_0) - \Lambda'(\sigma_0)(\sigma_1 - \sigma_0))g, g\rangle \leq \left\|\frac{\sigma_1}{\sigma_0} - 1\right\|_{L^\infty(\Omega)} \|g\|_\star^2.$$

Proof. The first inequality follows from Lemma 2.1. Furthermore, again by Lemma 2.1, we have

$$\begin{aligned}
&\sup_{\|g\|_\star=1} \langle(\Lambda(\sigma_1) - \Lambda(\sigma_0) - \Lambda'(\sigma_0)(\sigma_1 - \sigma_0))\, g, g\rangle \\
&\leq \sup_{\|g\|_\star=1} \left\langle \Lambda'(\sigma_0)\left(\left(\frac{\sigma_0}{\sigma_1} - 1\right)(\sigma_1 - \sigma_0)\right) g, g \right\rangle \\
&= \sup_{\|g\|_\star=1} \int_\Omega \left(\left(\frac{\sigma_1}{\sigma_0} - 1\right)\frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)\right)|\nabla u_0|^2 \, \mathrm{d}x \\
&\leq \left\|\frac{\sigma_1}{\sigma_0} - 1\right\|_{L^\infty(\Omega)}. \qquad \square
\end{aligned}$$

Finally, we show that bounds on the supremum in (4.3) guarantee the right outer support and that minimizing (4.3) improves the reconstructed values on the outer support.

LEMMA 4.2. *Assume that κ is piecewise-analytic and that there exist constants $0 \leq \delta_1 < \inf(\sigma_0/\sigma_1)$ and $\delta_2 \geq 0$ such that for all $g \in L^2_\diamond(\partial\Omega)$*

$$-\delta_1 \|g\|_\star^2 \leq \langle(\Lambda(\sigma_1) - \Lambda(\sigma_0) - \Lambda'(\sigma_0)\kappa)g, g\rangle \leq \delta_2 \|g\|_\star^2. \tag{4.4}$$

Then $\operatorname{supp}_{\partial\Omega} \kappa = \operatorname{supp}_{\partial\Omega}(\sigma_1 - \sigma_0)$, *and on* $\operatorname{supp}_{\partial\Omega} \kappa$

$$\left(\frac{\sigma_0}{\sigma_1} - \delta_1\right)(\sigma_1 - \sigma_0) \leq \kappa \leq (1 + \delta_2)(\sigma_1 - \sigma_0) \tag{4.5}$$

holds for all boundary points as described in Theorem 3.4.

Proof. For all $g \in L^2_\diamond(\partial\Omega)$, it follows from the lower bound in (4.4) and Lemma 2.1 that

$$\int_\Omega \left(\frac{\sigma_0}{\sigma_1}(\sigma_0 - \sigma_1) + \kappa\right) |\nabla u_0|^2 \, \mathrm{d}x \geq \langle(\Lambda(\sigma_1) - \Lambda(\sigma_0) - \Lambda'(\sigma_0)\kappa)g, g\rangle$$
$$\geq -\delta_1 \|g\|_\star^2 \geq -\delta_1 \int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \, \mathrm{d}x$$

and hence

$$\int_\Omega \kappa|\nabla u_0|^2 \, \mathrm{d}x \geq \int_\Omega \left(\frac{\sigma_0}{\sigma_1} - \delta_1\right)(\sigma_1 - \sigma_0)|\nabla u_0|^2 \, \mathrm{d}x. \tag{4.6}$$

Likewise, from the upper bound in (4.4) and Lemma 2.1, we obtain

$$\langle(\Lambda'(\sigma_0)(\sigma_1 - \sigma_0) - \Lambda'(\sigma_0)\kappa)\, g, g\rangle \leq \langle(\Lambda(\sigma_1) - \Lambda(\sigma_0) - \Lambda'(\sigma_0)\kappa)g, g\rangle$$
$$\leq \delta_2 \|g\|_\star^2 \leq \delta_2 \int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \, \mathrm{d}x,$$

and thus

$$\int_\Omega \kappa|\nabla u_0|^2 \, \mathrm{d}x \leq (\delta_2 + 1)\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \, \mathrm{d}x. \tag{4.7}$$

Using (4.6) and (4.7), the assertion follows from Theorem 3.4. ◻

We summarize the consequences of this section in the following remark.

Remark 4.3. Assume that we are given exact (infinite-dimensional) data

$$\Lambda(\sigma_0), \Lambda(\sigma_1) \in \mathcal{L}(L^2_\diamond(\Omega))$$

and that we can exactly calculate the Fréchet derivative

$$\Lambda'(\sigma_0) \in \mathcal{L}(L^\infty(\Omega); \mathcal{L}(L^2_\diamond(\Omega))).$$

By Lemma 4.1, there exists a sequence of piecewise-analytic functions κ_k so that

$$-\delta_1^{(k)} \leq \sup_{\|g\|_\star = 1} \langle(\Lambda(\sigma_1) - \Lambda(\sigma_0) - \Lambda'(\sigma_0)\kappa)g, g\rangle \leq \delta_2^{(k)} \tag{4.8}$$

with $\delta_1^{(k)}, \delta_2^{(k)} \in \mathbb{R}$ and $\delta_1^{(k)} \to 0$.

By Lemma 4.2 every such sequence κ_k will after finitely many steps (more precisely, if $\delta_1^{(k)} < \inf(\sigma_0/\sigma_1)$) possess the right outer support

$$\mathrm{supp}_{\partial\Omega}\, \kappa_k = \mathrm{supp}_{\partial\Omega}(\sigma_1 - \sigma_0)$$

and fulfill the bounds in (4.5). The bounds improve with smaller $\delta_1^{(k)}$ and $\delta_2^{(k)}$.

4.2. Noisy finite-dimensional data. In practice, we can attach only a limited number of electrodes through which we inject currents (Neumann data) and measure voltages (Dirichlet data). This means that we will have access only to approximations to $\Lambda(\sigma_1)$, $\Lambda(\sigma_0)$, and $\Lambda'(\sigma_0)$ (and thus to $\|\cdot\|_*$), and we can evaluate the supremum in (4.8) only over a finite-dimensional subspace V of $L^2_\diamond(\partial\Omega)$. Also, measurement errors are unavoidable.

The naive approach to using our method is to replace all operators by their approximations and minimize the bounds in (4.8) with the supremum taken only over V. We will show that this already yields convergence of the method if the dimensionality of V is chosen only so high that all expressions make sense. The latter may be interpreted as a *regularization by discretization*.

To be precise, we assume that we are given the following:

- $\Lambda_m(\sigma_0), \Lambda_m(\sigma_1) \in \mathcal{L}(L^2_\diamond(\Omega))$, $m \in \mathbb{N}$, with

$$\Lambda_m(\sigma_0) \to \Lambda(\sigma_0), \quad \Lambda_m(\sigma_1) \to \Lambda(\sigma_1) \quad \text{in } \mathcal{L}(L^2_\diamond(\Omega)). \tag{4.9}$$

- $\Lambda'_m(\sigma_0) \in \mathcal{L}(L^\infty(\Omega); \mathcal{L}(L^2_\diamond(\Omega)))$, $m \in \mathbb{N}$, with

$$\Lambda'_m(\sigma_0) \to \Lambda'(\sigma_0) \quad \text{in } \mathcal{L}(L^\infty(\Omega); \mathcal{L}(L^2_\diamond(\Omega))). \tag{4.10}$$

- Finite-dimensional subspaces V_k, $k \in \mathbb{N}$, with

$$V_1 \subset V_2 \subset \cdots \subset L^2_\diamond(\partial\Omega)$$

  and dense union in $L^2_\diamond(\partial\Omega)$.

We then equip V_k with the norm (cf. Theorem 4.4 below)

$$\|g\|^2_{\star(m)} := \langle (\Lambda_m(\sigma_0) - \Lambda_m(\sigma_1))g, g\rangle \tag{4.11}$$

and seek κ_k such that it minimizes bounds on the (Galerkin approximation of the) linearization residual

$$\Lambda_m(\sigma_1) - \Lambda_m(\sigma_0) - \Lambda'_m(\sigma_0)\kappa_k$$

in the sense of quadratic forms on V_k.

THEOREM 4.4.

(a) *Let $0 < \delta_1$ and $\delta_2 > \sup(\sigma_1/\sigma_0 - 1)$. For each V_k and sufficiently large m, $\|\cdot\|_{\star(m)}$ is a norm on V_k and $\kappa := \sigma_1 - \sigma_0$ fulfills*

$$-\delta_1 \leq \Lambda_m(\sigma_1) - \Lambda_m(\sigma_0) - \Lambda'_m(\sigma_0)\kappa \leq \delta_2 \tag{4.12}$$

in the sense of quadratic forms on V_k equipped with the $\|\cdot\|_{\star(m)}$-norm.

(b) *Let $(\kappa_k)_{k\in\mathbb{N}} \subset L^\infty(\Omega)$ be a sequence fulfilling* (4.12) *with constants* $0 < \delta_1 < \inf(\sigma_0/\sigma_1)$, $\delta_2 > 0$, *and with $m = m(k) \to \infty$ for $k \to \infty$. Then every piecewise-analytic limit κ of an (L^∞-)converging subsequence of κ_k fulfills*

$$\operatorname{supp}_{\partial\Omega} \kappa = \operatorname{supp}_{\partial\Omega}(\sigma_1 - \sigma_0),$$

and on $\operatorname{supp}_{\partial\Omega} \kappa$

$$\left(\frac{\sigma_0}{\sigma_1} - \delta_1\right)(\sigma_1 - \sigma_0) \leq \kappa \leq (\delta_2 + 1)(\sigma_1 - \sigma_0) \tag{4.13}$$

holds for all boundary points as described in Theorem 3.4.

Proof. (a) Since V_k is finite-dimensional there exists $c > 0$ such that

$$\|g\|^2_* = \langle (\Lambda(\sigma_0) - \Lambda(\sigma_1))g, g\rangle \geq c\|g\|^2_{L^2_\diamond(\partial\Omega)} \quad \text{for all } g \in V_k. \tag{4.14}$$

From (4.14) and $\Lambda_m(\sigma_0) - \Lambda_m(\sigma_1) \to \Lambda(\sigma_0) - \Lambda(\sigma_1)$, we conclude that

$$\|g\|^2_{\star(m)} \geq (1 - \epsilon_m)\|g\|^2_\star, \quad \text{with } \epsilon_m \to 0 \text{ for } m \to \infty. \tag{4.15}$$

In particular, this shows that $\|\cdot\|_{\star(m)}$ is a norm on V_k for sufficiently large m.

We will now show that $\sigma_1 - \sigma_0$ fulfills (4.12). From Lemma 4.1 and the convergence assumptions, assumptions (4.9), (4.10), it follows that there exists another zero sequence ϵ'_m with

$$\begin{aligned} -\epsilon'_m \|g\|^2_{L^2_\diamond(\partial\Omega)} &\leq \langle (\Lambda_m(\sigma_1) - \Lambda_m(\sigma_0) - \Lambda'_m(\sigma_0)(\sigma_1 - \sigma_0))g, g\rangle \\ &\leq \left\| \frac{\sigma_1}{\sigma_0} - 1 \right\|_{L^\infty(\Omega)} \|g\|^2_\star + \epsilon'_m \|g\|^2_{L^2_\diamond(\partial\Omega)} \end{aligned}$$

for all $g \in V_k$. Combining this with (4.14) and (4.15), we obtain (4.12) (for sufficiently large m).

(b) Now let κ be piecewise-analytic and (w.l.o.g.) $\|\kappa - \kappa_k\|_{L^\infty(\Omega)} \to 0$, where $(\kappa_k)_{k\in\mathbb{N}} \subset L^\infty(\Omega)$ is a sequence fulfilling (4.12) in V_k with $m = m(k) \to \infty$. For each fixed $g \in V_{k_0}$ we have by construction that

$$-\delta_1 \|g\|^2_{\star(m)} \leq \langle (\Lambda_m(\sigma_1) - \Lambda_m(\sigma_0) - \Lambda'_m(\sigma_0)\kappa_k)\, g, g\rangle \leq \delta_2 \|g\|^2_{\star(m)}$$

holds for a sequence $m(k) \to \infty$, $k \geq k_0$. Hence,

$$-\delta_1 \|g\|^2_\star \leq \langle (\Lambda(\sigma_1) - \Lambda(\sigma_0) - \Lambda'(\sigma_0)\kappa)\, g, g\rangle \leq \delta_2 \|g\|^2_\star. \tag{4.16}$$

By denseness and continuity, (4.16) holds for all $g \in L^2_\diamond(\partial\Omega)$. Hence we obtain the assertion from Lemma 4.2. ∎

Theorem 4.4 leaves open how to ensure the existence of a converging subsequence in part (b). We finish this work by describing one possibility based on additional assumptions. The following remark contains a one-step linearization algorithm whose reconstructions globally converge toward the right shape in the limit of exact data.

Remark 4.5. Consider the realistic case that we seek the conductivity difference $\sigma_1 - \sigma_0$ in some finite-dimensional ansatz space Σ of piecewise-analytic functions and that a bound $C > \sup \sigma_1$ is known. Furthermore assume that $\sigma_1 - \sigma_0$ belongs to the ansatz space Σ.

By Theorem 4.4(a), for sufficiently large m there exist solutions $\kappa_k \in \Sigma$ of

$$-\frac{\inf \sigma_0}{C} \leq -\delta_1^{(k)} < \Lambda_m(\sigma_1) - \Lambda_m(\sigma_0) - \Lambda'_m(\sigma_0)\kappa_k < \delta_2^{(k)} \leq \frac{C}{\inf \sigma_0} - 1 \tag{4.17}$$

with $\sup |\kappa_k| \leq C + \sup \sigma_0$, and again (4.17) is understood in the sense of quadratic forms on V_k equipped with the $\|\cdot\|_{\star(m)}$-norm from (4.11).

By the finite-dimensionality of Σ, every thus constructed sequence κ_k with $m(k) \to \infty$ will have an L^∞-converging subsequence with piecewise-analytic limit $\kappa \in \Sigma$. By Theorem 4.4(b), κ has the correct outer support and fulfills the bounds in (4.13). The bounds improve with smaller $\delta_1^{(k)}$ and $\delta_2^{(k)}$, i.e., with smaller linearization residual.

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