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## Autoencoder-based global concave optimization for electrical impedance tomography

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(joint work with Andrej Brojatsch, Johannes Wagner)

We report on some preliminary work-in-progress results that aim to derive globally convergent reconstruction algorithms for the inverse coefficient problem of Electrical Impedance Tomography (aka the famous Calderón problem) with finitely-many measurements.

**The Calderón problem with finitely-many measurements.** Let $\Omega \subseteq \mathbb{R}^k$, $k \geq 2$ be a bounded domain with smooth boundary $\partial\Omega$. Let $\sigma \in L^\infty_+(\Omega)$ and let

$$\Lambda(\sigma) : \ L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u_\sigma^{(g)}|_{\partial\Omega}$$

be the Neumann-to-Dirichlet-operator (aka current-to-voltage map) for the EIT equation, i.e., $u_\sigma^{(g)} \in H^1_\diamond(\Omega)$ solves

$$\nabla \cdot (\sigma \nabla u_\sigma^{(g)}) = 0 \quad \text{in } \Omega, \quad \sigma \partial_\nu u_\sigma^{(g)}|_{\partial\Omega} = g.$$

It is easily shown that $\Lambda(\sigma) \in \mathcal{L}(L^2_\diamond(\partial\Omega))$ is a compact and selfadjoint operator.

The inverse problem

$$\text{reconstruct} \quad \sigma \in L^\infty_+(\Omega) \quad \text{from} \quad \Lambda(\sigma) \in \mathcal{L}(L^2_\diamond(\partial\Omega))$$

has become famous under the name *Calderón problem*. It is known to be a highly non-linear and ill-posed problem. To introduce its variant with finitely many measurements, we introduce a pixel partition

$$\overline{\Omega} = \bigcup_{j=1}^n \overline{P_j}$$

where $P_1, \ldots, P_n \subseteq \Omega$ are non-empty, pairwise disjoint subdomains with Lipschitz boundaries. We assume that the coductivity coefficient $\sigma \in L^\infty_+(\Omega)$ is piecewise constant with respect to this partition, i.e. $\sigma = \sum_{j=1}^n \sigma_j \chi_{P_j}$, with $\sigma_1, \ldots, \sigma_n \in \mathbb{R}_+$, and $\chi_{P_j}$ denoting the characteristic function on the $j$-th pixel. With a slight abuse of notation, we identify a piecewise constant function $\sigma \in L^\infty_+(\Omega)$ with the vector

$\sigma = (\sigma_1, \ldots, \sigma_n)^T \in \mathbb{R}^n_+$. As a model for finitely-many measurements, we assume that we can measure the symmetric matrix

$$F(\sigma) = \left( \int_{\partial\Omega} g_i \Lambda(\sigma) g_j \mathrm{d}s \right)_{i,j=1,\ldots,m} \in \mathbb{S}^m \subset \mathbb{R}^{m \times m}$$

for $m$ given boundary currents $g_1, \ldots, g_m \in L^2_\diamond(\partial\Omega)$. This corresponds to measuring the Galerkin projection of $\Lambda(\sigma)$ to the span of $g_1, \ldots, g_m$. The gap electrode model in EIT can be written in this form by choosing $g_j$ to be the characteristic function of the $j$-th electrode, and more sophisticated electrode models such as the shunt model or the complete electrode model lead to similar properties of $F$. We can thus state the *Calderón problem with finitely many measurements*

$$\text{reconstruct} \quad \sigma \in \mathbb{R}^n_+ \quad \text{from} \quad F(\sigma) \in \mathbb{S}^m \subset \mathbb{R}^{m \times m}.$$

**Concave data fitting formulation for EIT.** Standard data-fitting formulations for EIT lead to non-convex minimization problems in high dimensions for which globally convergent algorithms may seem completely out-of-reach. However, the recent result [2] shows that it is possible to reformulate the Calderón problem (with sufficiently many measurements and known a-priori bounds on $\sigma$) as a convex semidefinite optimization problem. The reformulation involves an unknown linear cost functional so that its practical implementation is not immediate. We herein use a different (and simpler) approach to formulate the problem as a concave minimization problem over a convex set.

**Lemma 1.** *The following holds:*

*(a) If $\sigma \in \mathbb{R}^n_+$ fulfills $F(\sigma) = Y$ then $\sigma$ minimizes*

$$\operatorname{trace}(Y - F(\sigma)) \to \min! \quad \textit{s.t.} \quad F(\sigma) \preceq Y.$$

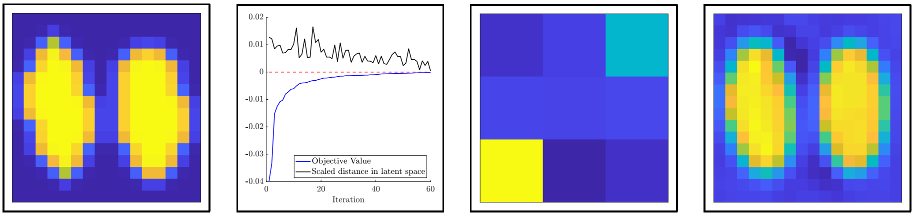
*(b) The functional $\sigma \mapsto \operatorname{trace}(Y - F(\sigma))$ is concave.*
*(c) The constraint set $\{\sigma \in \mathbb{R}^n_+ : \ F(\sigma) \preceq Y\}$ is convex.*

*Proof.* This follows from the fact that $F : \ \mathbb{R}^n_+ \to \mathbb{S}^m$ is monotonically non-increasing and convex with respect to the componentwise ordering "$\leq$" on $\mathbb{R}^n$ and the Loewner ordering "$\preceq$" on $\mathbb{S}^m$, cf. [2, Lemma 4.7]. □

In practical applications one usually also knows a-priori upper and lower bounds of $\sigma$ so that the constraint set becomes convex and bounded.

**Globally convergent concave programming.** Concave optimization problems over convex bounded sets can be solved with globally convergent algorithms in moderately low dimensions, cf. [3], and [5, Chp. 7.2]. The key idea is that global minima of concave functionals on a polyhedra are attained in a corner. Thus, for a bounded convex set, one starts with a polyhedron containing the constraint set, finds the best corner, i.e. the global minimizer on this superset, and then cuts out the best corner with a hyperplane to shrink the polyhedron. This approach should also yields global convergence for our concave minimization problem in Lemma 1.

FIGURE 1. Concave minimization in the latent space.




Ground Truth | Iteration Plot | Latent Variable | Reconstruction

**Concavity preserving autoencoder parametrization.** As concave global minimization is numerically feasible in moderately low dimensions, we now aim to describe our unknown conductivies by moderately many parameters. In the work [4], autoencoder techniques were used to find a 16-dimensional latent parametrization of lung images. The key idea is to train neural networks $\Phi$ and $\Psi$ so that

$$\Psi \circ \Phi \approx \mathrm{id} \quad \text{ on training set of lung images,}$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}^d$ *encodes* $n$-pixel images with $d$ latent parameters, and $\Psi : \mathbb{R}^d \to \mathbb{R}^n$ *decodes* $n$-pixel images from $d$ latent parameters. To solve the inverse problem $F(\sigma) = Y$ for a lung image $\sigma$, one then solves $F(\Psi(p)) = Y$ for $p \in \mathbb{R}^d$, and obtains $\sigma = \Psi(p)$.

Combining this idea with our concave optimization approach in Lemma 1 we would thus minimize

$$\operatorname{trace}(Y - F(\Psi(p))) \to \min! \quad \text{s.t.} \quad F(\Psi(p)) \preceq Y.$$

It is easily shown that this a concave minimization problem on a convex set, if the decoder $\Psi$ is concave. Since the decoder is a neural network that is a concatenation of linear functions and activator functions, it can be ensured to be convex by enforcing these linear functions and activator functions to be convex and non-decreasing. Training the autoencoder with shifted negated images one can thus construct a concave decoder $\Psi$.

Hence, we can reconstruct the conductivity by concave minimization in the low-dimensional latent parameter space. Figure 1 shows a preliminary numerical result for this approach using a 9-dimensional latent variable space and a FEM-implemenation of the EIT forward problem with $m = 31$ electrodes following [1]. The first image shows the true lung image, and the second image the error of the iterations (black line), and the (appropriately scaled) value of the objective functional (blue line). The third image shows the reconstructed latent variable $p \in \mathbb{R}^9$ as a $3 \times 3$-image, and the last image shows the reconstructed lung image $\sigma = \Psi(p)$. Note that the objective functional converges monotonically to zero from below as the iterates are the global minimizers of the concave objective functional on a polygonal superset of the constraint set.

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# Learned iterative reconstructions in photoacoustic tomography for the acoustic and optical problem


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## 1. Learned Reconstructions

We consider the general form of the underlying operator equation

$$A(f) = g, \tag{1}$$

for the inverse problem with $f \in X$ and $g \in Y$. Here, the forward operator $A\colon X \to Y$ can be either linear, for the acoustic problem, or the nonlinear for the optical problem. For the modelling of the forward problem in photoacoustic tomograpgy we refer to [3].

In the following we concentrate on the inverse problem, which can be understood as formulating a reconstruction operator $\mathcal{R} : Y \to X$. Such a reconstruction operator should be ideally stable and provide a good estimate of the original signal $f$ for given data $g$, i.e., $\mathcal{R}(g) \approx f$. Classically, such a reconstruction operator would be handcrafted based on the analytical knowledge of the forward operator, or formulated in the variational framework as optimisation problem.

In recent years, the paradigm of data-driven reconstructions has gathered considerable attention, due to its success in improving reconstruction quality, but also computational speed-up. Nevertheless, the majority of such data-driven approaches still comes without a thorough mathematical understanding. While we can not solve this shortcoming, we will provide a conceptual overview of data-driven approaches in the following. For that, let first us define the concept of a learned reconstruction operator.

**Definition 1** (Learned reconstruction operator)**.** *A family of mappings $\mathcal{R}_\theta\colon Y \to X$ parametrised by $\theta \in \Theta$ is called a* learned reconstruction operator *for the inverse problem in* (1) *if the parameters $\theta$ are determined (learned) from example data (training data) that is generated in a way that is consistent with* (1)*.*